The Potential of a Homogeneous Sphere $|\bar{x}| \le R$

Radius R

$$\Delta V = -4\pi \rho \qquad \rho = \frac{M}{\frac{4}{3}R^3}$$

M = mass.

$$\rho = 0 \text{ outside}$$

$$V(\bar{x}) = \iiint_{|\bar{y}| \le R} \frac{\rho(\bar{y})\, d^3\bar{y}}{|\bar{x} - \bar{y}|}$$

(Newtonian Potential)

The integral can be evaluated*) but it is simpler to solve the radial equations

*) Integrate first with respect to θ in spherical coordinates

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} = -4\pi\rho$$

$$\underbrace{r^2 \frac{\partial V}{\partial r^2} + 2r\frac{\partial V}{\partial r}}_{\frac{d}{dr}\left(r^2 \frac{\partial V}{\partial r}\right)} = -4\pi\rho r^2$$

$$r^2 \frac{\partial V}{\partial r} = -\frac{4}{3}\pi\rho r^3 + C_1$$

$$\frac{\partial V}{\partial r} = -\frac{4}{3}\pi\rho r - \frac{C_1}{r^2}$$

$$V = -\frac{1}{2}\frac{4}{3}\pi\rho r^2 + \frac{C_1}{r} + C_2$$

$$V = \begin{cases} -\frac{1}{2}\cdot\frac{4}{3}\pi\rho r^2 + C_2, & r \le R \\ \frac{C_1}{r} + C_2', & r \ge R \end{cases}$$

The term ($\frac{C_1}{r}$ cannot be present inside the ball)

By adding a constant to the potential we can take $C_2' = 0$

Continuity at $r = R$ requires

$$\begin{cases} -\frac{1}{2R} M + C_2 = \frac{C_1}{R} & \text{for } V \\ -\frac{M}{R^2} = -\frac{C_1}{R^2} & \text{for } \frac{\partial V}{\partial r} \end{cases}$$

Thus

$$C_1 = M, \quad C_2 = \frac{3}{2}\frac{M}{R}$$

$$V(r) = \begin{cases} \frac{M}{2R^3}(3R^2 - r^2), & 0 \leq r \leq R \\ \frac{M}{r}, & r \geq R \end{cases}$$

The gravitational force $\bar{F} = \gamma \nabla V$ is

$$\begin{cases} -\frac{\gamma M}{r^2}\bar{e}_r & \text{outside } (r \geq R) \\ -\frac{\gamma M r}{R^3}\bar{e}_r & \text{inside } (r \leq R) \end{cases} \qquad \bar{e}_r = \frac{\vec{x}}{|\bar{x}|}$$