

9.3/18 (The solutions to the boundary value problem $\Delta u = -f$ are unique. A minimizer is a solution. Then the minimizer is the unique solution!)

$$u(x, y) = \frac{1}{2} (-xy + xy^2 + x^2y - x^2y^2)$$

$$= -\frac{1}{2} xy(1-y)(1-x)$$

$$-\Delta u = \underbrace{x^2 + y^2 - x - y}_{\equiv f} = -[x(1-x) + y(1-y)]$$

Boundary values

$$I(u) = \iint_{0,0}^{1,1} \left(\frac{|\nabla u|^2}{2} - fu \right) dx dy, \quad u|_{\partial Q} = 0$$

The above u is a minimizer.

$$\frac{\partial u}{\partial x} = \dots = \frac{1}{2} y(1-y)(2x-1), \quad \frac{\partial u}{\partial y} = \frac{1}{2} x(1-x)(2y-1)$$

$$|\nabla u|^2 = \frac{1}{4} y^2(1-y)^2(2x-1)^2 + \frac{1}{4} x^2(1-x)^2(2y-1)^2$$

$$\left(\int_0^1 (2x-1)^2 dx = \frac{1}{2} \int_0^1 (2x-1)^3 = \frac{1}{3}, \quad \int_0^1 y^2(1-y)^2 dy = \frac{1}{30} \right)$$

$$\boxed{\frac{1}{2} \iint_{0,0}^{1,1} |\nabla u|^2 dx = \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{30}}$$

$$\{u = \frac{1}{2} x^2(1-x)^2 y(1-y) + \frac{1}{2} y^2(1-y)^2 x(1-x)$$

$$\boxed{\iint_{0,0}^{1,1} fu dx dy = \frac{1}{6 \cdot 30}}$$

$$I(u) = \frac{1}{4 \cdot 3 \cdot 30} - \frac{1}{6 \cdot 30} = -\frac{1}{360}$$

Ex.: $u=0, I(u)=0$ Ex.: $u=\min(x\pi) \min(y\pi), I(u)=?$
 (Keep f the same all the time.) Long calculation.

$$9.3/1) \quad \begin{cases} \Delta u = -f \text{ in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

$$I(u) = \iint_{\Omega} \left(\frac{1}{2} |\nabla u|^2 - fu \right) dx dy \leq I(0) = 0$$

since $v \equiv 0$ is admissible.

If $I(u) = 0$, then also $I(v) = \min I(u)$.

But then 0 satisfies the Euler-Lagrange eqn:

$$\Delta v = -f, \quad \Delta v = 0$$

But by assumption $f \neq 0$.

b) No! If u_0 is the above solution, then $v = u + C$ (a constant) is a solution with boundary values C and

$$I(v) = I(u_0) - C \iint_{\Omega} f dx dy$$

If $\iint_{\Omega} f dx dy \neq 0$, we have a positive value for $|C|$ large enough. (We skip the case $\iint_{\Omega} f dx dy = 0$.)

9.3/20 (ii), (iv) have wrong boundary values.

9.3/25 The functions $u_k(x) = \frac{x^k}{k}$ ($k = 1, 2, 3, \dots$)

are admissible. $Q[u_k] = \frac{1}{2^{k-1}} \rightarrow 0$. Thus

$\inf_u Q[u] = 0$. A function with $Q[u] = 0$ must be constant. No constant is admissible here!

