Notes on Sobolev Spaces

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1 L^p-SPACES

1.1 Inequalities

For any measurable function $u : A \to [-\infty, \infty],\ A \in \mathbb{R}^n$, we define

$$\|u\|_p = \|u\|_{p,A} = \left\{ \int_A |u(x)|^p \, dx \right\}^{\frac{1}{p}}$$

and, if this quantity is finite, we say that $u \in L^p(A)$. In most cases of interest $p \geq 1$. For $p = \infty$ we set

$$\|u\|_\infty = \|u\|_{\infty,A} = \operatorname*{ess\,sup}_{x\in A} |u(x)|.$$

The essential supremum is the smallest number M such that $|u(x)| \leq M$ for a.e. $x \in A$.

For example[1], if $u : [a, b] \to \mathbb{R}^n$, is continuous, then it is easy to see that

$$\left\{ \int_a^b |u(x)|^p \, dx \right\}^{\frac{1}{p}} \underset{p\to\infty}{\longrightarrow} \max_{a\leq x\leq b} |u(x)|.$$

The fundamental inequalities

$$\|uv\|_1 \leq \|u\|_p \|v\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1, \qquad \text{HÖLDER}$$

$$\|u + v\|_p \leq \|u\|_p + \|v\|_p, \qquad \text{MINKOWSKI}$$

[1] An interesting application of this fact in connection with the heat equation $u_t = u_{xx}$ is given in [BB, Ch.24, pp 145-150].

where $p \geq 1$, can be derived in many ways. For example, taking the "elementary inequality"

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad (a, b \geq 0,\ p + q = pq) \quad \text{YOUNG}$$

for granted, we obtain the Hölder inequality (choose $a = u(x)/||u||_p$, $b = v(x)/||u||_q$ and integrate the resulting inequality). The Hölder inequality implies the Minkowski inequality.

Remarks:

1) The special case

$$\int_A |uv|\, dx \leq \sqrt{\int_A u^2\, dx}\sqrt{\int_A v^2\, dx} \qquad \text{CAUCHY}$$

is called the Cauchy inequality.

2) If $1 > p > 0$, then the Minkowski inequality is reversed for positive functions! That is why one usually has $p \geq 1$.

3) If mes $(A) < \infty$, then the Hölder inequality shows that $L^{p_1}(A) \subset L^{p_2}(A)$, if $p_1 \geq p_2$. This is not true in general, if mes $(A) = \infty$. (Find a simple ex.!) However, if $1 \leq p_1 < p < p_2 \leq \infty$, then the Hölder inequality implies that

$$||u||_{p,A} \leq ||u||^{\lambda}_{p_1,A} ||u||^{1-\lambda}_{p_2,A}$$

where

$$\frac{1}{p} = \frac{\lambda}{p_1} + \frac{1-\lambda}{p_2}.$$

That is, if $u \in L^{p_1} \cap L^{p_2}$, then $u \in L^p$ for all intermediate p.

4) Suppose that $0 <$ mes $(A) < \infty$. The function

$$\phi(p) = \left\{ \frac{1}{\text{mes}\,(A)} \int_A |u(x)|^p\, dx \right\}^{\frac{1}{p}}, \quad 0 < |p| < \infty,$$

$$\phi(0) = \exp\left\{ \frac{1}{\text{mes}\,(A)} \int_A \log |u(x)|\, dx \right\},$$

$$\phi(-\infty) = \operatorname{ess\,inf}_A |u(x)|, \quad \phi(+\infty) = \operatorname{ess\,sup}_A |u(x)|$$

is *increasing*[2] as $-\infty \leqq p \leqq +\infty$.

Let us finally mention that

$$\Lambda\left(\frac{1}{\text{mes}\,(A)}\int\limits_A u(x)\,dx\right) \leq \frac{1}{\text{mes}\,(A)}\int\limits_A \Lambda(u(x))\,dx \qquad \text{JENSEN}$$

whenever $\Lambda : \mathbb{R} \to \mathbb{R}$ is a convex function ($0 < \text{mes}\,(A) < \infty$, $\quad u : A \to [-\infty, \infty]$ is measurable).

Example:

$$e^{\int_0^1 u(x)\,dx} \leq \int\limits_0^1 e^{u(x)}\,dx$$

$$\left\{\frac{1}{\text{mes}\,(A)}\int\limits_A |u(x)|\,dx\right\}^p \leq \frac{1}{\text{mes}\,(A)}\int\limits_A |u(x)|^p\,dx$$

Remark: Discrete versions of the above inequalities are

$$\left|\sum_k a_k b_k\right| \leq \sqrt[p]{\sum_k |a_k|^p}\,\sqrt[q]{\sum_k |b_k|^q};$$

$$n\,a_1 a_2 \ldots a_n \leq |a_1|^n + |a_2|^n + \ldots + |a_n|^n$$

or

$$\underbrace{\sqrt[n]{q_1 q_2 \ldots q_n}}_{\text{GEOMETRIC MEAN}} \leq \underbrace{\frac{q_1 + q_2 + \ldots + q_n}{n}}_{\text{ARITHMETIC MEAN}} \quad (q_k \geq 0);$$

$$\sqrt[p]{\sum_k |a_k + b_k|^p} \leq \sqrt[p]{\sum_k |a_k|^p} + \sqrt[p]{\sum_k |a_k|^p}.$$

Especially, $\lim\limits_{p\to\infty} \sqrt[p]{\sum_k |a_k|^p} = \sup\limits_k |a_k|$.

Example: Draw the curves $\sqrt[p]{|x|^p + |y|^p} = 1 \quad (0 < p \leqq \infty)$ in the xy-plane.

[2] J. Moser's celebrated method to relate the maximum and the minimum of the solution to a partial differential equations is based on this.

For $p = 2$ we have the parallelogram[3] law

$$\|u+v\|_2^2 + \|u-v\|_2^2 = 2\|u\|_2^2 + 2\|v\|_2^2.$$

Remark: So-called reverse Hölder inequalities like

$$\left(\frac{1}{\text{mes }(Q)}\int\limits_Q |u|\right)^{\frac{1}{p}} \le \frac{C}{\text{mes }(Q)}\int\limits_Q |u|$$

valid for *every* cube with some fixed $p > 1$ (!) play a central role for solutions u to certain partial differential equations. (Such an inequality cannot hold for arbitrary functions.) The method is due to F. W. GEHRING.

1.2 (Strong) convergence in L^p

Let $A \in \mathbb{R}^n$ and fix $p \ge 1$. Then $\|\ \|_p$ defines a *semi-norm* in the vector space $L^p(A)$, i.e.,

(i)' $0 \le \|u\|_p < \infty, \quad u \in L^p(A)$

(ii) $\|\lambda u\|_p = |\lambda|\|u\|_p, \quad u \in L^p(A),\ -\infty < \lambda < \infty$

(iii) $\|u+v\|_p \le \|u\|_p + \|v\|_p, \quad u, v \in L^p(A).$

However, this is, strictly speaking, not a norm, the reason being that $\|u\|_{p,A} = 0 \Longleftrightarrow u(x) = 0$ for a.e. $x \in A$. We agree to say that $u = v$ in $L^p(A)$, if $u(x) = v(x)$ for a.e. $x \in A$. With this convention $\|\ \|_p$ is a *norm,* i.e. (i)' can be replaced by

(i) $0 \le \|u\|_p < \infty$, and $\|u\|_p = 0 \Longleftrightarrow u = 0$ in L^p.

(Strictly speaking, this L^p-space consists of *equivalence classes of functions,* but here there is no point in maintaining this distinction.)

Theorem 1 (RIESZ-FISCHER) *The L^p-spaces, $p \ge 1$, are Banach spaces. That is, if $u_1, u_2, \ldots$ is a Cauchy sequence in $L^p(A)$, then there is a function $u \in L^p(A)$ such that $\|u_k - u\|_{p,A} \to 0$, as $k \to \infty$.*

[3]The counterpart for general exponents p is more involved. A pair of inequalities (CLARKSON's inequalities) will replace the parallelogram law [A, p.37].

Proof: Suppose that $u_1, u_2, \ldots$ is a Cauchy sequence in $L^p(A)$, i.e., given $\varepsilon > 0$ there is an index N_ε such that $\|u_k - u_j\|_p < \varepsilon$, when $k, j \geq N_\varepsilon$. Let us consider the case $p \neq \infty$ (the case $p = \infty$ is simpler, and its proof will be skipped). We can construct indices $1 \leq n_1 < n_2 < \ldots$ such that

$$\|u_{n_k} - u_{n_{k+1}}\|_p < \frac{1}{2^k} \quad (k = 1, 2, 3, \ldots).$$

Define

$$g_N(x) = \sum_{k=1}^{N} |u_{n_k} - u_{n_{k+1}}|, \quad g(x) = \lim_{N\to\infty} g_N(x) = \sum_{k=1}^{\infty} |u_{n_k} - u_{n_{k+1}}|$$

Then

$$0 \leq g_1(x) \leq g_2(x) \leq \ldots \leq g(x) \leqq \infty$$

and

$$\int_A |g|^p = \int_A \left|\lim_{N\to\infty} g_N\right|^p = \int_A \lim_{N\to\infty} |g_N|^p \leq \underline{\lim}_{N\to\infty} \int_A |g_N|^p$$

by Fatou's lemma. By the construction $\|g_N\|_p \leq \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^N} < 1$ and so $\int_A |g|^p \leq 1$. Thus $g(x) < +\infty$ for a.e. $x \in A$. This means that the series $|u_{n_2}(x) - u_{n_1}(x)| + |u_{n_3}(x) - u_{n_2}(x)| + \ldots$ converges for a.e. $x \in A$. So does *a fortiori,* the series

$$u_{n_2}(x) + \sum_{k=1}^{\infty} (u_{n_{k+1}} - u_{n_k}).$$

The partial sums of this series are plainly $u_{n_2}(x), u_{n_3}(x), u_{n_4}(x), \ldots$ Hence

$$u(x) = \lim_{k\to\infty} u_{n_k}(x)$$

exists and is finite for a.e. $x \in A$ (If you insist on having an everywhere defined function, set u(x)=0 in a subset of measure zero.)

By Fatou's lemma (again!)

$$\int_A |u(x) - u_j(x)|^p \, dx = \int_A \left(\lim_{k\to\infty} |u_{n_k}(x) - u_j(x)|^p\right) dx \leq \underline{\lim}_{k\to\infty} \int_A |u_{n_k}(x) - u_j(x)|^p \, dx \leq \varepsilon^p$$

whenever $j > N_\varepsilon$. This shows that $\|u - u_j\|_{p,A} \to 0$ as $j \to \infty$. □

The proof yields information about pointwise behaviour.

Corollary 2 *Suppose that $u_k \to u$ in $L^p(A)$, i.e., $\|u_k - u\|_{p,A} \to 0$. Then there is a subsequence that converges a.e. in A : $u(x) = \lim_{k\to\infty} u_{n_k}(x)$ for a.e. $x \in A$.*

1.3 The dual of L^p

If X is any Banach space, its *dual* X^* is the collection of all continuous linear functions (functionals) $l : X \to \mathbb{R}$. The norm of l is defined as

$$||l||_{X^*} = \sup_{x\in X} \left(\frac{|l(x)|}{||x||_X} \right) = \sup_{||x||_X \leq 1} |l(x)|$$

and the continuity is equivalent to $||l||_{X^*} < \infty$. Note that

$$|l(x)| \leq ||l||_{X^*} ||x||.$$

Let $1 \leq p < \infty$ and fix a function $g \in L^q(A)$. Then

$$l(f) = \int_A f(x)g(x)\,dx \quad (f \in L^p(A))$$

is well-defined and linear. By Hölder's inequality $|l(f)| \leq ||f||_p ||g||_q$. Hence $||l||_* \leq ||g||_q$. Here equality is attained for the choice $f = |q|^{q-2}g$. (The case $q = \infty$ is slightly different.) Thus

$$||l||_* = ||g||_q.$$

The essential fact here is that virtually *all* continuous functionals in L^p $(p \neq \infty)$ are of this form!

Theorem 3 (F. RIESZ' representation thm) *Let* $l : L^p(A) \to \mathbb{R}$ *be a continuous linear functional,* $1 \leq p < \infty$. *Then there is a unique* $g \in L^q(A)$ *such that*

$$l(f) = \int_A f(x)g(x)\,dx$$

for all $f \in L^p(A)$. *Moreover,* $||l||_* = ||g||_{q,A}$.

Proof: The Radon-Nikodym theorem is used in most proofs of the representation theorem. A more direct proof in the one dimensional case is given in [R. p.121]. □

Remark:

1) There are continuous linear functionals in $L^\infty(A)$ that do not have this simple form.

2) One says that L^q *is the dual of* L^p $(1 \leq p < \infty)$.

1.4 Weak convergence in L^p

Weak convergence in L^p is important for many applications, for example, it leads to the existence theorems for partial differential equations. The so-called direct methods in the Calculus of Variations are based on this concept.

Suppose that $u_1, u_2, u_3, \dots$ are functions in L^p. *We say that $u_k \to u$ weakly in L^p, if*

$$\lim_{k\to\infty} l(u_k) = l(u)$$

for every continuous linear functional $l : L^p \to \mathbb{R}$. Using Riesz' representation theorem we can state the definition in a more convenient form.

Definition 4 *Let $1 \le p < \infty$. Suppose that $u_1, u_2, \dots$ and u are functions in $L^p(A)$. We say that $u_k \to u$ weakly in $L^p(A)$, if*

$$\lim_{k\to\infty} \int_A u_k v \, dx = \int_A uv \, dx \quad \textit{for each } v \in L^q(A).$$

Remark:

1) Sometimes the notation $u_k \rightharpoonup u$ is used to indicate weak convergence.

2) Strong convergence implies weak convergence: if $\|u_k - u\|_p \to 0$, then $u_k \rightharpoonup u$ weakly in L^p.

3) The weak limit is unique, that is, unique in L^p.

Example: The functions $u_n(x) = \sin(nx)$ converge *weakly* in $L^2([0, 2\pi])$ to zero. By the Riemann-Lebesgue lemma

$$\lim_{n\to\infty} \int_0^{2\pi} v(x) \sin(nx) \, dx = 0$$

for every v in $L^2([0, 2\pi])$. (This follows easily from Bessel's inequality.)

Example: (Warning!) Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded measurable function, for example, assume $0 \le f(x) \le 2$. Then there is a sequence of functions $v_n : [a, b] \to \{0, 2\}$ converging weakly in $L^2([a, b])$ to f. Observe that $v_n(x) = 0$ or $= 2$ (v_n takes no other values!!)
This is not difficult to realise in the special case $f(x) \equiv 1$.

Example: Define $u_i :]0,1[\to \mathbb{R}$ by $u_i = i\chi_{]0,i^{-3}[}$ for $i = 1, 2, 3, \ldots$ Then $u_i \to 0$ strongly in $L^p([0,1])$, if $1 \leq p < 3$. We have $\|u_i\|_{3+\varepsilon} = i^{\varepsilon/3}$ for every $\varepsilon > 0$. As we shall see, weakly convergent sequences are bounded in L^p-norm. Therefore $u_1, u_2, \ldots$ does not converge weakly in $L^p(]0,1[)$, if $p > 3$. —Show that $u_i \rightharpoonup 0$ weakly in $L^3(]0,1[)$! (This convergence is not strong.)

If $u_i \rightharpoonup u$ weakly in $L^p(A)$, then there is a constant M such that $\|u_i\|_{p,A} \leq M \leq \infty$ for all $i = 1, 2, 3, \ldots$ This follows from the following lemma.

Lemma 5 *Suppose that $u_1, u_2, \ldots$ are functions in $L^p(A)$, $1 < p < \infty$. If the sequence $\|u_i\|_{p,A}$ is unbounded, there is a $w \in L^q(A)$ such that*

$$\lim_{k\to\infty} \int_A u_{i_k}(x) w(x)\, dx = +\infty$$

for some subsequence.

Proof: The function w is constructed and written down in [S, pp.25-28]. □

The most important fact about L^p-spaces seems to be the following *weak compactness* property. (*Not* valid for $p = 1$).

Theorem 6 *(Weak Compactness) Let $u_1, u_2, \ldots$ be functions in $L^p(A)$, $1 < p < \infty$. If there is a constant M such that $\|u_i\|_{p,A} \leq M$ for each index i, then there exists a function $u \in L^p(A)$ such that $u_{i_k} \rightharpoonup u$ weakly in $L^p(A)$ for some subsequence. Moreover*

$$\|u_i\|_{p,A} \leq \varliminf_{k\to\infty} \|u_{i_k}\|_{p,A}.$$

Proof: Advanced books on Functional Analysis usually contain a proof. For example, the above mentioned book of Sobolev [S] gives a proof on pp. 29-30. See also [A]. Usually, the lower semicontinuity comes as a by-product of the proof, but the following simple argument also yields this property. Since $|x|^p$ is a convex function of x, we have

$$|y|^p \geq |x|^p + p|x|^{p-2}x \cdot (y - x), \quad p \geq 1.$$

Thus

$$\int_A |u_{i_k}|^p\, dx \geq \int_A |u|^p\, dx + p \int_A |u|^{p-1}u \cdot (u_{i_k} - u)\, dx.$$

Now $|u|^{p-1}u$ is in $L^q(A)$, and so the last integral approaches zero as $k \to \infty$ (by the weak convergence!). This gives us the desired lower semicontinuity. □

Remarks:

1) *The theorem is not true for* $p = 1$.

2) The existence of solutions to partial differential equations is often a direct consequence of the weak compactness.

3) Let $\|u_i\|_p \leq M$ for $i = 1, 2, 3, \dots$ According to the BANACH-SAKS theorem there are indices $i_1 < i_2 < i_3 < \dots$and a function u in L^p such that

$$\left\| \frac{u_{i_1} + u_{i_2} + \dots + u_{i_\nu}}{\nu} - u \right\|_p \longrightarrow 0$$

as $\nu \to +\infty$, that is, the *arithmetic means converge strongly.* (The Banach-Saks theorem is valid also for $p = 1$.)

1.5 Approximation in L^p and some other things

The essential fact is that the functions in L^p can be approximated in the L^p-norm by smooth functions, if $1 \leq p < \infty$. These are constructed as convolutions. Let us first state some auxiliary results.

Lemma 7 ("Continuity in the L^p-norm") *Let* $1 \leq p < \infty$. *If* $f \in L^p(A)$, *then*

$$\lim_{h \to 0} \int_A |f(x+h) - f(x)|^p \, dx = 0,$$

where f *is regarded as* 0 *outside* A.

The *proof* is not quite simple. Usually one uses the theorem of Lusin, valid for measurable functions.

Theorem 8 (Lusin) *Let* $f : A \to [-\infty, \infty]$ *be a measurable function that is finite a.e. in* A. *Suppose that* $(A) < \infty$. *Given* $\varepsilon > 0$, *there is a compact set* $K_\varepsilon \subset A$ *such that the restriction* $f|_{K_\varepsilon}$ *is continuous and* $mes(A \setminus K_\varepsilon) < \varepsilon$.

Proof: See, [EG, p.15].

If A is *open,* then Lusin's theorem can be combined with the extension theorem of Urysohn-Tietze [R, p.148].

Theorem 9 (Urysohn-Tietze) *Suppose that A is open and mes $(A) < \infty$. Let $K \subset A$ be a compact set. If $f : K \to \mathbb{R}$ is a continuous function, then there is a function $\varphi \in C_0(A)$ such that $\varphi(x) = f(x)$, when $x \in K$. Moreover,* $\max_A|\varphi| = \max_A|f|$

Lemma 10 *$C_0(\Omega)$ is dense in $L^p(\Omega)$, $1 \le p < \infty$. Here Ω is open and $C_0(\Omega)$ denotes all continuous functions with compact support on Ω. In other words, if $u \in L^p(\Omega)$, then there are functions $\varphi_k \in C_0(\Omega)$ such that*

$$\lim_{k\to\infty}\|u - \varphi_k\|_{p,\Omega} = 0.$$

Remarks:

1) Of course, this implies that $C(\Omega)$ is dense in $L^p(\Omega)$.

2) Functions in $L^\infty(\Omega)$ cannot, in general, be uniformly approximated by continuous functions.

3) If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuous, then the *closure* in $\mathbb{R}^n$ of the set where $\varphi(x) \neq 0$ is called the *support* of φ. Thus

$$\operatorname{supp}(\varphi) = \overline{\{x \in \mathbb{R}^n \,|\varphi(x) \neq 0\}}.$$

If $\operatorname{supp}(\varphi)$ is compact (it is closed by definition) and if $\operatorname{supp}(\varphi) \subset \Omega$, then we say that $\varphi \in C_0(\Omega)$. In this case the distance between $\operatorname{supp}(\varphi)$ and the boundary $\partial\Omega$ is positive.

4) A deep result for L^p-functions is related to the Lebesgue points. Suppose that $f \in L^p_{\text{loc}}(\mathbb{R}^n)$, that is, $f \in L^p(B)$ for every ball B in $\mathbb{R}^n$. Then

$$\lim_{r\to o^+}\frac{1}{\text{mes }(B(x,r))}\int\limits_{B(x,r)} |f(x) - f(y)|^p\, dy = 0$$

for a.e. x.

Define

$$\rho(x) = \begin{cases} Ce^{-\frac{1}{1-|x|^2}}, & |x| < 1 \\ 0, & |x| \ge 1 \end{cases}$$

and choose the constant $C > 0$ such that $\int_{\mathbb{R}^n} \rho(x)\, dx = 1$. "Friedrichs' mollifier" is

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho\left(\frac{x}{\varepsilon}\right) = \begin{cases} \dfrac{C}{\varepsilon^n}e^{-\frac{\varepsilon^2}{\varepsilon^2-|x|^2}}, & \text{when } |x| < \varepsilon, \\ 0, & \text{when } |x| \ge \varepsilon\,. \end{cases}$$

The constant C depends only on the dimension n. Now we have

$$\int_{\mathbb{R}^n} \rho_\varepsilon(x)\,dx = 1, \quad \varepsilon > 0. \tag{1}$$

Observe that $\rho_\varepsilon \in C_0^\infty(\mathbb{R}^n)$. *The support of* ρ_ε *is the closed ball* $|x| \leq \varepsilon$.

The *convolution*

$$u_\varepsilon(x) = (\rho_\varepsilon * u)(x) = \int_{\mathbb{R}^n} \rho_\varepsilon(x-y)u(y)\,dy$$

is well-defined for $u \in L^1_{\text{loc}}(\mathbb{R}^n)$. If u is defined only in the domain Ω, then we regard u as extended to zero outside $\Omega : u(x) = 0$, when $x \in \mathbb{R}^n \setminus \Omega$. Hence we can calulate the convolution u_ε for any u in $L^1_{\text{loc}}(\Omega)$.

Observe that u_ε *is always a smooth function:* $u_\varepsilon \in C^\infty(\mathbb{R}^n)$, if $u \in L^1_{\text{loc}}(\mathbb{R}^n)$. For differentiation we have the rule

$$D^\alpha u_\varepsilon = (D^\alpha \rho_\varepsilon) * u. \tag{2}$$

Here $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and

$$D^\alpha = \frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}.$$

If u was defined in the domain Ω, then the formula (2) holds for the original (unextended) u at all points x with $dist(x, \partial\Omega) > \varepsilon$. Analogously, if supp $u \subset \Omega$, then $u_\varepsilon \in C_0^\infty(\Omega)$, when $\varepsilon < dist(\text{supp}\, u, \partial\Omega)$.

Lemma 11 *If* $u \in L^p(\Omega)$, $1 \leq p < \infty$, *then* $u_\varepsilon \in L^p(\Omega)$ *and* $\|\rho_\varepsilon * u\|_p \leq \|u\|_p$. *Moreover*

$$\lim_{\varepsilon \to 0^+} \|u_\varepsilon - u\|_p = 0.$$

Lemma 12 *If* $u \in C(\Omega)$ *and if* $K \subset\subset \Omega$ *is compact then*

$$\max_{x \in K} |u_\varepsilon(x) - u(x)| \to 0 \quad \textit{as} \quad \varepsilon \to 0^+.$$

In other words, the convergence $u_\varepsilon \to u$ *is uniform on compact subsets.*

Proof: The locally uniform convergence for a continuous u follows form

$$\begin{aligned} u(x) - u_\varepsilon(x) &= u(x) - \int \rho_\varepsilon(x-y)u(y)\,dy \\ &= \int\limits_{|x-y|<\varepsilon} (u(x)-u(y))\rho_\varepsilon(x-y)\,dy \\ |u(x) - u_\varepsilon(x)| &\le \max_{\substack{y \\ |x-y|\leqq\varepsilon}} |u(x)-u(y)| \cdot 1 \quad (\varepsilon < dist(x, \partial\Omega)) \end{aligned}$$

in accordance with Weierstrass theorem (a function that is continuous on a compact set us uniformly continuous).

If $u \in L^p(\Omega)$, then

$$|u_\varepsilon| \le \int \rho_\varepsilon(x-y)|u(y)|\,dy \le \left\{\int \rho_\varepsilon(x-y)|u(y)|^p\,dy\right\}^{\frac{1}{p}} \underbrace{\left\{\int \rho_\varepsilon(x-y)\,dy\right\}^{\frac{1}{q}}}_{=1}$$

and so

$$\begin{aligned} \int |u_\varepsilon(x)|^p\,dx &\le \int \left(\int \rho_\varepsilon(x-y)|u(y)|^p\,dy\right) dx \\ &= \int |u(y)|^p \underbrace{\left(\int \rho_\varepsilon(x-y)\,dx\right)}_{=1} dy = \int |u(y)|^p\,dy. \end{aligned}$$

This proves the contraction $\|u_\varepsilon\|_p \le \|u\|_p$.

For the L^p-convergence, we first note that if $\varphi \in C_0(\Omega)$, then it is easily seen that

$$\int |\varphi(x) - \varphi_\varepsilon(x)|^p\,dx \le \sup_{|x-y|\leqq\varepsilon} |\varphi(x) - \varphi(y)|^p \quad (\varepsilon < dist(\operatorname{supp}\varphi, \partial\Omega))$$

and so $\|\varphi - \varphi_\varepsilon\|_p \to 0$. (Doing some more work, we could replace C_0 by $C \cap L^p$.) Now

$$\|u - u_\varepsilon\|_p \le \|u - \varphi\|_p + \|\varphi - \varphi_\varepsilon\|_p + \underbrace{\|\overbrace{\varphi_\varepsilon - u_\varepsilon}^{=(\varphi-u)_\varepsilon}\|_p}_{\le\|\varphi-u\|_p}$$

and so

$$\overline{\lim_{\varepsilon\to0^+}} \|u - u_\varepsilon\|_p \le 2\|u - \varphi\|_p.$$

Since $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$, we can choose $\varphi \in C_0(\Omega)$ so that $\|u - \varphi\|_p$ is as small as we please. This concludes the proof for the L^p-convergence $\|u - u_\varepsilon\|_p \to 0$. □

As an application we mention the Variational Lemma.

Lemma 13 (Variational Lemma) *Let $u \in L^1_{loc}(\Omega)$, Ω denoting an open set in $\mathbb{R}^n$. If*

$$\int_\Omega u(x)\varphi(x)\,dx = 0$$

whenever $\varphi \in C_0^\infty(\Omega)$, then $u = 0$ a.e. in Ω.

Proof: Take $\bar{x} \in \Omega$ and choose ε so small that $0 < \varepsilon < dist(\bar{x}, \partial\Omega)$. Then $\rho_\varepsilon(\bar{x} - x)$ will do as a test function so that

$$u_\varepsilon(\bar{x}) = \int u(x)\rho_\varepsilon(\bar{x} - x)\,dx = 0.$$

Let B be any closed ball in Ω, i.e. $B \subset\subset \Omega$. Then

$$\|u\|_{1,B} = \|u - u_\varepsilon\|_{1,B} \to 0$$

as $\varepsilon \to 0^+$. Hence $u = 0$ a.e. in B and hence a.e. also in Ω. □

Remark: The lemma is fundamental in the Calculus of Variations. If u is continuous, the proof is more elementary and $u \equiv 0$ in this case. We shall need the lemma to establish that the Sobolev derivatives of a function are unique up to sets of measure zero.

Example: (Hermann WEYL) Let $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ and suppose that

$$\int u(x)\,\Delta\varphi(x)\,dx = 0$$

whenever $\varphi \in C_0^\infty(\mathbb{R}^n)$. Then u has a continuous representative which is a harmonic function, i.e. the continuous u belongs to C^2 and $\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} = 0$. (In fact, $u \in C^\infty(\mathbb{R}^n)$.)

Take $\varphi(x) = \rho_\varepsilon(x - z)$ for some fixed z. Then $\Delta u_\varepsilon(z) = 0$. Indeed,

$$u_\varepsilon(z) = (\rho_\varepsilon * u)(z) = \int \rho_\varepsilon(z - x)\,u(x)\,dx$$

$$\Delta u_\varepsilon(z) = (\Delta\rho_\varepsilon * u)(z) = \int \Delta_z\rho_\varepsilon(z - x)\,u(x)\,dx$$

$$= \int [\Delta_x\rho_\varepsilon(z - x)]u(x)\,dx = 0$$

by assumption. Note that

$$\frac{\partial^2 u}{\partial x_k^2}\rho_\varepsilon(z-x) = \frac{\partial^2 u}{\partial z_k^2}\rho_\varepsilon(z-x)$$

by direct calculation ($\rho_\varepsilon(z-x)$ is a function of $|z-x|^2$). Since z was arbitrary, $\Delta u_\varepsilon \equiv 0$. Thus the function u_ε is harmonic.

One does not seem to get any further without using some deeper property of harmonic functions. By the mean value property

$$u_\varepsilon(x) = \frac{1}{\text{mes }(B(x,r))}\int\limits_{B(x,r)} u_\varepsilon(y)\,dy \quad (\text{Actually } u_\varepsilon = u!)$$

Since $u_\varepsilon \to u$ in $L^1(B(x,r))$, we have

$$\lim_{\varepsilon\to 0^+} u_\varepsilon(x) = \frac{1}{\text{mes }(B(x,r))}\int\limits_{B(x,r)} u(y)\,dy.$$

For a.e. x, $u(x) = \lim u_\varepsilon(x)$, at least when ε approaches zero through a subsequence $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ (Corollary 2). Redefining u in a set of measure zero, we get a function that satifies the mean value property. Hence (the redefined) u is harmonic in $\mathbb{R}^n$ $\square$.

2 SOBOLEV SPACES

The situation with the derivatives belonging to some L^p-space was studied by Tonelli, B. Levi, Sobolev, Kondrachev et consortes. The corresponding spaces are named after Sobolev.

2.1 $W^{m,p}$ and $H^{m,p}$

Throughout this chapter Ω denotes a domain or an open subset of $\mathbb{R}^n$. Suppose $u \in C^1(\Omega)$, where Ω. Then integration by parts yields

$$\int_\Omega u(x)\frac{\partial \varphi}{\partial x_k}\,dx = -\int_\Omega \frac{\partial u}{\partial x_k}\varphi(x)\,dx$$

when $\varphi \in C_0^\infty(\Omega)$. This formula is the starting point for the definition of *weak* (distributional, generalized) *partial derivatives.*

Definition 14 *Assume $u \in \mathrm{L}^1_{loc}(\Omega)$. We say that $v_k \in \mathrm{L}^1_{loc}(\Omega)$ is the weak partial derivative of u with respect to x_k in Ω if*

$$\int_\Omega u(x)\frac{\partial \varphi}{\partial x_k}\,dx = -\int_\Omega v_k(x)\varphi(x)\,dx$$

for all $\varphi \in C_0^1(\Omega)$. We write $v_k = D_k u = \frac{\partial u}{\partial x_k}$ and $\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right)$, provided the weak derivatives exists.

The weak partial derivative is uniquely defined a.e. in Ω (by the Variational lemma). Notice the requirement that the weak derivative is a *function* (to which Lebesgue's theory applies), not merely a distribution. It is sufficient to consider all $\varphi \in C_0^\infty(\Omega)$ in the integration-by-parts formula.

We say that the function u belongs to the SOBOLEV SPACE $W^{1,p}(\Omega)$, if $u \in L^p(\Omega)$ and the weak partial derivatives

$$\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n}$$

exist and belong to $\mathrm{L}^p(\Omega)$. Here $1 \le p \le \infty$. We say that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ if $u \in W^{1,p}(U)$ for each open $U \subset\subset \Omega$. (In this case we may have $\|u\|_{p,\Omega} = \infty$

or $\|\nabla u\|_{p,\Omega} = \infty$.) For $u \in W^{1,p}(\Omega)$ we define the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{1,p,\Omega} = \|u\|_{p,\Omega} + \sum_{k=1}^{n} \left\| \frac{\partial u}{\partial x_k} \right\|_{p,\Omega}$$

Any equivalent norm, as

$$\|u\| = \left\{ \int_\Omega [|u|^p + |\nabla u|^p] \; dx \right\}^{\frac{1}{p}},$$

will do. We say that $u_k \to u$ (strongly) in $W^{1,p}(\Omega)$, if $\|u_k - u\|_{W^{1,p}(\Omega)} \to 0$. Provided with this norm (or any equivalent norm) $W^{1,p}(\Omega)$ *is a Banach space.*

Example: Let $u(x) = |x|$ for $-\infty < x < \infty$. It is easily verified that

$$v(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

is the weak derivative of u. Show that $\int \varphi v \, dx = -\int u\varphi' \, dx$! (We can set $v(0) = 0$ or $v(0) = A$ or even $v(0) = \infty$. Sets of measure zero do not count.)

Example: Let $v(x) = 1$, when $x \geq 0$ and $v(x) = -1$, when $x < 0$. Then the weak derivative of v does not exist in $(-2, 2)$, for example. The origin is the crucial point. (Dirac's delta is not a *function.*)

Higher weak derivatives are defined in a similar way. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ are multi-indices, we write

$$|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n, \quad \alpha! = \alpha_1!\alpha_2!\ldots\alpha_n!$$

$$\binom{\alpha}{\beta} = \frac{\alpha!}{(\alpha-\beta)!\beta!} = \prod_{k=1}^{n} \binom{\alpha_k}{\beta_k} \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$$

$$D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n} = \frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}$$

In this notation

$$D^\alpha(\varphi\psi) = \sum_{0 \leqq |\beta| \leqq |\alpha|} \binom{\alpha}{\beta} D^{\alpha-\beta}\varphi \cdot D^\beta \psi, \quad \text{LEIBNIZ' RULE}$$

$$\varphi(x+h) = \varphi(x) + \sum_{0<|\alpha|} D^\alpha \varphi(x) \cdot \frac{h^\alpha}{\alpha!}. \quad \text{TAYLOR'S FORMULA}$$

If $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ and $v_\alpha \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ are related by

$$\int_\Omega u(x)\, D^\alpha \varphi(x)\, dx = (-1)^{|\alpha|} \int_\Omega v_\alpha(x)\varphi(x)\, dx$$

for all $\varphi \in C_0^\infty(\Omega)$, we write $v_\alpha = D^\alpha u$. This is a weak derivative of order $|\alpha|$.

We say that $u \in \mathrm{W}^{\mathrm{m,p}}(\Omega)$, if $u \in \mathrm{L}^{\mathrm{p}}(\Omega)$ and if $D^\alpha u$ exists and belongs to $\mathrm{L}^{\mathrm{p}}(\Omega)$ for each multi-index α with $|\alpha| \le k$. Provided with the norm

$$\|u\|_{\mathrm{W}^{\mathrm{m,p}}(\Omega)} = \|u\|_{m,p,\Omega} = \sum_{|\alpha| \le m} \left\{ \int_\Omega |D^\alpha u|^p\, dx \right\}^{\frac{1}{p}},$$

the Sobolev space $\mathrm{W}^{\mathrm{m,p}}(\Omega)$ is a Banach space. The term with index $\alpha = (0, 0, \ldots, 0)$ is interpreted as $\left\{ \int_\Omega |u|^p\, dx \right\}^{\frac{1}{p}}$. The integer m counts the order of the highest weak derivative.

Theorem 15 *Assume that $u \in \mathrm{W}^{\mathrm{m,p}}(\Omega)$ for some $1 \le p < \infty$. Then there exists a sequence of functions $\varphi_k \in C^\infty(\Omega)$ such that $\varphi_k \to u$ in $\mathrm{W}^{\mathrm{m,p}}(\Omega)$, i.e.,*

$$\|u - \varphi_k\|_{\mathrm{W}^{\mathrm{m,p}}(\Omega)} \overset{k\to\infty}{\to} 0.$$

About the *proof.* The case $\Omega = \mathbb{R}^n$ is relatively simple. The general case was is in [A, pp.52-53]. Let us just mention that the proof[4] uses the partition of unity.

Partition of unity

The partition of unity is frequently used in the theory of distributions. Suppose that $\Omega \subset \bigcup_{j=1}^{\infty} U_j$ where each U_j is open. Then there are functions $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that:

1) $0 \le \varphi \le 1$.

2) $\varphi_1(x) + \varphi_2(x) + \ldots = 1$ at each point x in Ω.

3) If $K \subset \Omega$ is any compact set, then only finitely many of the functions φ_k are not identically zero in K.

[4]This was proved by N. Meyers and J. Serrin in 1964. However, this was not, as it were, the first proof.

4) Each $\varphi_j \in C_0^\infty(U_k)$ for some $k = k(j)$.

We define $\mathrm{H^{m,p}}(\Omega)$ *as the completion of* $C^\infty(\Omega)$ *in the norm* $\|\ \|_{m,p,\Omega}$. Often H^m means $H^{m,2}$, $p = 2$ being the most important special case. To be more precise, $u \in \mathrm{L^p}(\Omega)$ belongs to the space $\mathrm{H^{m,p}}(\Omega)$, if there are functions $\varphi_i \in C^\infty(\Omega)$ such that $\varphi_i \to u$ in $\mathrm{L^p}(\Omega)$ and $D^\alpha\varphi_i$ is a Cauchy sequence in $\mathrm{L^p}(\Omega)$ for each multi-index α, $|\alpha| \le m$.

It is not difficult to see that $D^\alpha u$ exists, $|\alpha| \le m$, if $u \in \mathrm{H^{m,p}}(\Omega)$. Moreover, $D^\alpha\varphi_i \to D^\alpha\varphi$ in $\mathrm{L^p}(\Omega)$. The central result is [A, pp.52-53]:

Theorem 16 $\mathrm{H^{m,p}}(\Omega) = \mathrm{W^{m,p}}(\Omega), \quad 1 \le p < \infty, \quad m = 1, 2, 3, \dots$

There is also a characterization of the Sobolev space in terms of *integrated difference quotients.* To this end, let $e_i = (0, \dots, 1, \dots, 0)$ denote the unit vector in the i^{th} direction. If $u \in W^{1,p}(\Omega)$, *then*

$$\int_{\Omega'} \left| \frac{u(x + he_i) - u(x)}{h} \right|^p dx \le \int_{\Omega} |D_i u|^p\, dx$$

for any subdomain $\Omega' \subset\subset \Omega$, *when* $0 < h < dist(\Omega', \partial\,\Omega)$.
(For smooth functions φ this follows from the identity

$$\frac{\varphi(x + he_i) - \varphi(x)}{h} = \frac{1}{h} \int_0^h D_i\varphi(x_1, \dots, x_i + t, \dots, x_n)\, dt$$

and the general case follows by approximation.)

Remark: For smooth functions in *convex* domains the formula

$$\varphi(y) - \varphi(x) = \int_0^1 \left[\frac{d}{dt}\varphi(x + t(y - x)) \right] dt = \int_0^1 (y - x) \cdot \nabla\varphi(x + t(y - x))\, dt$$

is the source of many useful inequalities.

Theorem 17 *Let* $u \in \mathrm{L^p}(\Omega), \quad 1 < p < \infty$. *Suppose that for any subdomain* $\Omega' \subset\subset \Omega$ *we have*

$$\int_{\Omega'} \left| \frac{u(x + he_i) - u(x)}{h} \right|^p dx \le K^p < \infty$$

whenever $0 < h < dist(\Omega', \partial\,\Omega)$. *Then the weak derivative* $D_i u$ *exists and* $\|D_i u\|_{p,\Omega} \le K$.

Proof: See [GT, p.169].

In conclusion, there are three[5] equivalent definitions for the Sobolev space:

I The definition based on the *integration-by-parts formula.*

II The definition based on *approximation by smooth functions* with respect to the Sobolev norm.

III The characterization in terms of the integrability of *difference quotients.*

Often, II is used to prove auxiliary inequalities and imbedding theorems, III is sometimes used to prove the existence of weak derivatives. But I is the Main Definition.

When we say that $u \in W^{1,p}(\Omega)$ is, for example, continuous, we mean that there exists a continuous function φ such that $u(x) = \varphi(x)$ for a.e. $x \in \Omega$. Thus u can be made continuous after a redefinition in a set of measure zero. (Remember that, in general, functions in L^p are defined only almost everywhere.)

2.2 The Space $W_0^{1,p}(\Omega)$

We wish to introduce functions with boundary values zero in Sobolev's sense. Remember that functions in $W^{1,p}(\Omega)$ can be approximated by functions in $C^\infty(\Omega)$ with respect to the norm

$$\|u\|_{1,p,\Omega} = \|u\|_{p,\Omega} + \sum_{k=1}^{n} \left\| \frac{\partial u}{\partial x_k} \right\|.$$

If the approximation can be done using merely functions with compact support Ω, then the function itself is in a closed subspace denoted by $W_0^{1,p}(\Omega)$.

Definition 18 *Suppose that $u \in W^{1,p}(\Omega)$. We say that $u \in W_0^{1,p}(\Omega)$, if, given $\varepsilon > 0$, there is a function $\varphi_\varepsilon \in C_0^\infty(\Omega)$ such that $\|u - \varphi_\varepsilon\|_{1,p,\Omega} < \varepsilon$.*

Hence $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the corresponding Sobolev norm. Clearly,

$$C_0^p(\Omega) \subset W_0^{1,p}(\Omega) \subset W^{1,p}(\Omega).$$

[5] If Ω is the whole space $\mathbb{R}^n$, a further definition is possible. It is based on the Fourier transform. There is also a more advanced theory based on Bessel potentials.

There are functions in $\subset W_0^{1,p}(\Omega)$, that do not have compact support in Ω. (Example: $n = 1$, $p = 2$, $\Omega =]0,\pi[$, $u(x) = \sin x$, $\operatorname{supp} u = [0,\pi] \supset \Omega$.) If $u \in C(\Omega) \cap W^{1,p}(\Omega)$ and if

$$\lim_{\substack{x\to\xi \\ x\in\Omega}} u(x) = 0$$

at each boundary point $\xi \in \partial\Omega$, then $u \in W_0^{1,p}(\Omega)$. However, there are continuous functions in the Sobolev space $W_0^{1,p}(\Omega)$ that do not have "the right boundary values" zero in the classical sense:

Example: $\Omega = \{x \in \mathbb{R}^3 \,|\, 0 < |x| < 1\}$. Here $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Then

$$u(x) = \ln\frac{1}{|x|}, \quad x \in \Omega,$$

is in $W_0^{1,2}(\Omega)$. The origin is an isolated boundary point. We have

$$\lim_{x\to 0} \ln\frac{1}{|x|} = +\infty \quad (\neq 0).$$

(At all the other boundary points the function has the right boundary values in the classical sense.)

Remark: Let $\Omega \subset \mathbb{R}^n$ and suppose that $p > n$. Then every function in $W_0^{1,p}(\Omega)$ is continuous and takes the boundary values zero in the classical sense. (In applications, one usually has $p = 2 < 3 \leqq n$, unfortunately.)

An extremely important property of the space $W_0^{1,p}(\Omega)$ is that it is *closed even under weak convergence.* That is, if $u_1, u_2, \ldots$ belong to $W_0^{1,p}(\Omega)$ and if $u_i \rightharpoonup u$ and $\nabla u_i \rightharpoonup \nabla u$ weakly in $\mathrm{L}^p(\Omega)$ (by definition $u \in W^{1,p}(\Omega)$), then u itself is in $W_0^{1,p}(\Omega)$.

Let $f \in W^{1,p}(\Omega)$ and suppose that $u \in W^{1,p}(\Omega)$. We say that u has the boundary values f in Sobolev's sense, if $u - f \in W_0^{1,p}(\Omega)$. (Sometimes this is written as $u \in f + W_0^{1,p}(\Omega)$.) —There is a theory of so-called Trace Spaces like $W^{k,\alpha}(\partial\Omega)$ for general boundary value problems. In particular,the normal derivative $\frac{\partial u}{\partial n}$ makes sense.

Lemma 19 If $u \in W_0^{1,p}(\Omega)$, then the function $u\chi_\Omega + 0\chi_{\mathbb{R}^n\setminus\Omega}$ is in $W_0^{1,p}(\mathbb{R}^n)$.

If $\Omega \subset \mathbb{R}^n$, then the cases $1 \leq p < n$, $p = n$ (the borderline case), and $p > n$ are very different in some essential estimates.

Theorem 20 (the Sobolev inequality) *Suppose that $u \in W_0^{1,p}(\Omega)$, where Ω is a domain in $\mathbb{R}^n$. If $1 \leq p < n$, then there is a constant C depending only on n and p such that*

$$\left(\int_\Omega |u|^{\frac{np}{n-p}} \, dx \right)^{\frac{n-p}{np}} \leq C \left(\int_\Omega |\nabla u|^p \, dx \right)^{\frac{1}{p}}. \quad \textit{SOBOLEV}$$

Thus $u \in L^{p^*}(\Omega)$; $\quad p^* = \frac{np}{n-p}$ = the Sobolev conjugate.

Proof: Extending u as zero outside Ω we may assume that $u \in W_0^{1,p}(\mathbb{R}^n)$. By approximation it is sufficient to prove the inequality for $\varphi \in C_0^\infty(\mathbb{R}^n)$. Let us, for instructive purposes, write down the proof in the two dimensional case $n = 2, x = (x_1, x_2)$. Multiply

$$\varphi(x_1, x_2) = \int_{-\infty}^{x_1} \frac{\partial \varphi(t, x_2)}{\partial x_1} \, dt, \quad \varphi(x_1, x_2) = \int_{-\infty}^{x_2} \frac{\partial \varphi(x_1, t)}{\partial x_2} \, dt,$$

to get

$$|\varphi(x_1, x_2)|^2 \leq \int_{-\infty}^{\infty} \left| \frac{\partial \varphi(t_1, x_2)}{\partial x_1} \right| dt_1 \cdot \int_{-\infty}^{\infty} \left| \frac{\partial \varphi(x_1, t_2)}{\partial x_2} \right| dt_2.$$

Integrate with respect to x_1 :

$$\int_{-\infty}^{\infty} |\varphi(x_1, x_2)|^2 \, dx_1 \leq \int_{-\infty}^{\infty} \left| \frac{\partial \varphi(t_1, x_2)}{\partial x_1} \right| dt_1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |D_2 \varphi| \, dx_1 \, dx_2.$$

Integrate with respect to x_2 :

$$\iint |\varphi|^2 \, dx_1 \, dx_2 \leq \iint |D_2 \varphi| \, dx_1 \, dx_2 \cdot \iint |D_2 \varphi| \, dx_1 \, dx_2 \leq \left(\iint |\nabla \varphi| \, dx_1 \, dx_2 \right)^2.$$

Hence $\|\varphi\|_2 \leq \|\nabla \varphi\|_1$. This proves the theorem in the case $n = 2, \ p = 1$.

For the general p, $1 < p < 2$, set $\psi = |\varphi|^\gamma$ with $\gamma > 1$ as selected below. Then the above inequality applied on ψ yields

$$\iint |\varphi|^{2\gamma} \leq \gamma^2 \left(\iint |\varphi|^{\gamma - 1} |\nabla \varphi| \right)^2 \overset{\text{HÖLDER}}{\leq} \gamma^2 \left(\iint |\varphi|^{2\gamma} \right)^{2\frac{\gamma-1}{2\gamma}} \left(\iint |\nabla \varphi|^{\frac{2\gamma}{\gamma+1}} \right)^{\frac{\gamma+1}{\gamma}}$$

for $\gamma > 1$. Take $2\gamma = \frac{2p}{2-p}$. Then, after some arithmetic,

$$\left(\iint |\varphi|^{\frac{2p}{2-p}}\right)^{\frac{2-p}{p}} \leq \gamma^2 \left(\iint |\nabla\varphi|^p\right)^{\frac{2}{p}}.$$

This is the desired inequality for $1 < p < n = 2$.

See [EG, pp.138-140] or [GT, pp.155-156] for general n. A proof can also be based on the formula

$$u(x) = \frac{1}{\omega_n} \int_\Omega \frac{(x-y)\cdot\nabla u(y)}{|x-y|^n}\, dy$$

valid for $u \in W_0^{1,1}(\Omega)$. □

Remark: The case $n < p < \infty$. Let $u \in W^{1,p}(\Omega)$, $\quad p > n$. For each cube $Q \subset \Omega$ we have

$$|u(x) - u(y)| \leq \frac{2pn}{p-n} |x-y|^{1-\frac{n}{p}} \|\nabla u\|_{p,Q}$$

for a.e. $x, y \in Q$. Hence u is locally Hölder continuous in Ω with Hölder exponent $\alpha = 1 - \frac{n}{p}$. If Ω is bounded, then $u \in C(\bar{\Omega})$ and $\|u\|_{\infty,\Omega} \leq C|\,\Omega\,|^{1/n-1/p}\|\nabla u\|_{p,\Omega}$.

The case $p = n$ (borderline case[6]) is very special.

For functions in $W^{1,p}$ (but not in $W_0^{1,p}$) the corresponding inequalities are more involved than the Sobolev inequality and they do require some additional regularity of the domain in question (balls, cubes, domains, smooth domains, Lipschitz domains etc).

1) $1 < p < n, \quad u \in W^{1,p}(Q), \quad Q =$ a cube in $\mathbb{R}^n$. Then

$$\|u\|_{p^*,Q} \leq p^* \frac{n-1}{n} \|\nabla u\|_{p,Q} + |Q|^{-1/n} \|u\|_{p,Q}$$

(The gain is that $p^* = \frac{np}{n-p} > p$.)

2) *Poincaré's* inequality for $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$.

$$\int_\Omega |u(x) - u_\Omega|^p\, dx \leq C(n,p)|\,\Omega\,|^{p/n} \int_\Omega |\nabla u(x)|^p\, dx.$$

[6] The TRUDINGER-MOSER inequality

$$\int_\Omega \exp\left(\frac{|u|}{c\|\nabla u\|_n}\right)^{n/(n-1)} dx \leq C \operatorname{mes}(\Omega)$$

holds.

Here Ω is a convex domain and $u_\Omega = \frac{1}{|\Omega|}\int\limits_\Omega u(x)\,dx$ is the average of u over Ω.

3) *Poincaré-Sobolev*: $1 \le p < n, \quad u \in W^{1,p}(\Omega), \quad \Omega = \text{convex}.$

$$\left(\int\limits_\Omega |u(x) - u_\Omega|^{\frac{np}{n-p}}\,dx\right)^{\frac{n-p}{np}} \le C_{n,p}\left(\int\limits_\Omega |\nabla u(x)|^p\,dx\right)^{1/p}$$

These inequalities constitute the main bulk in the *Sobolev Imbedding Theorem.* For example, $L^q(\Omega) \supset W_0^{1,p}(\Omega)$, if $q = p^*$, and, for a domain of finite Lebesgue measure, this is valid if $1 \le q \le p^*$. The *Rellich-Kondrachev* theorem is even stronger.

Theorem 21 (Rellich[7]-Kondrachev) *Let Ω be an arbitrary bounded domain in $\mathbb{R}^n$ and consider the space $W_0^{1,p}(\Omega)$.*

<u>$1 \le p < n.$</u> *Then $W_0^{1,p}(\Omega)$ is compactly imbedded in $L^q(\Omega)$, where $1 \le q < p^* = \frac{np}{n-p}$ (q need not be conjugate to p). In practical terms, if $u_i \in W_0^{1,p}(\Omega)$, and*

$$\|u_i\|_{1,p,\Omega} \le M, \qquad i = 1, 2, \ldots,$$

then there exists a function $u \in W_0^{1,p}(\Omega)$ and a subsequence such that

$$\|u - u_i\|_{q,\Omega} \to 0 \quad as \quad i \to \infty$$

for each fixed q, $1 \le q < p^*$.

<u>$p > n.$</u> *Then $W_0^{1,p}(\Omega)$ is compactly imbedded in $C^\alpha(\bar{\Omega})$, $\alpha = 1 - \frac{n}{p}$. Now*

$$|u(x) - u(y)| \le K|x - y|^\alpha.$$

<u>$p = n.$</u> *This is the borderline case.*

The same is true for $W^{1,p}(\Omega)$, if the boundary of Ω is sufficiently regular.

Remarks:

- The functions, not their derivatives, converge *strongly* in $L^q(\Omega)$.
- The convergence $u_{i_\nu} \rightharpoonup u$ need not be strong in $L^{p^*}(\Omega)$, but $u_{i_\nu} \rightharpoonup u$ *weakly* also in $L^{p^*}(\Omega)$.

[7]The case $p = 2$ is credited to Rellich.

2.3 About $W^{1,p}$

The *first* order Sobolev space has some special properties not valid for higher derivatives. If $u \in W^{1,p}_{\text{loc}}(\Omega)$, so does u^+ and u^-. As usually, $u = u^+ - u^-$, $|u| = u^+ + u^-$. Hence, $|u| \in W^{1,p}_{\text{loc}}(\Omega)$. We have

$$\nabla u^+ = \begin{cases} \nabla u, & \text{when } u > 0, \\ 0, & \text{when } u \leqq 0, \end{cases}$$

with similar rules for u^- and $|u|$. For example, for $|u|$ we use

$$\int (u_\nu^2 + \varepsilon^2)^{1/2} \nabla\varphi \, dx = -\int \varphi \frac{u_\nu \nabla u_\nu}{(u_\nu^2 + \varepsilon^2)^{1/2}} \, dx \quad (\varepsilon \neq 0).$$

The passage to the limit under the integral sign can be justified [R, Ch.4,Thm.6]. Now

$$|\varphi u_\nu \nabla u_\nu|(u_\nu^1 + \varepsilon^2)^{-1/2} \leq |\varphi \nabla u_\nu|.$$

Finally, letting $\varepsilon \to 0$, we have by the Dominated Convergence Theorem

$$\int |u| \nabla\varphi \, dx = -\int \varphi \left[1\chi_{\{u>0\}} - 1\chi_{\{u<0\}} + 0\chi_{\{u=0\}} \right] \nabla u \, dx$$

i.e., $\nabla|u| = \nabla u$, when $u > 0$, $= -\nabla u$, when $u < 0$, and $= 0$, when $u = 0$.

Lemma 22 *Let $u \in W^{1,p}(\Omega)$. Then $\nabla u = 0$ a.e. on any set where u is constant.*

Lemma 23 $u, v \in W^{1,p}(\Omega) \Longrightarrow \max\{u, v\}$, $\min\{u, v\} \in W^{1,p}(\Omega)$.

The last lemma is not true with $W^{1,p}$ replaced by $W^{k,p}$, if $k \geq 2$. Example!?

3 The equation $\Delta u = 4\alpha u^3 + f(x), \quad \alpha \geq 0$

Let Ω be a *bounded* domain in $\mathbb{R}^n$. Suppose that $f \in L^\infty(\Omega)$ and that $\alpha \geq 0$ is , for simplicity, a constant. Fix a function $\varphi \in W^{1,2}(\Omega)$. Usually, φ is continuous even in $\bar{\Omega}$. It represents the boundary values.

Problem: *Minimize the variational integral*

$$I(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 + \alpha u^4 + f(x)u \right] dx$$

among all $u \in W^{1,2}(\Omega)$ with boundary values φ (in Sobolev's sense, i.e., $u - \varphi \in W_0^{1,2}(\Omega)$).

The problem is interesting even for the case $\varphi \equiv 0$. Before proving the existence of a unique solution, let us observe that

[I]

The function $u \in W^{1,2}(\Omega)$, $u - \varphi \in W_0^{1,2}(\Omega)$, *is minimizing if and only if*

$$\int \left[\nabla u \cdot \nabla \eta + 4\alpha u^3 \eta + f(x)\eta\right] dx = 0 \quad \text{EULER-LAGRANGE EQN in weak form}$$

for all test-functions $\eta \in W_0^{1,2}(\Omega)$. (Hence a partial integration and the Variational lemma leads to the Eqn

$$\Delta u = 4\alpha u^3 + f(x),$$

provided that u has a second derivatives.)

To see this, note that for any real number ε

$$I(u + \varepsilon\eta) = I(u) + \varepsilon \int [\nabla u \cdot \nabla \eta + 4\alpha u^3 \eta + f(x)\eta]\, dx + \frac{1}{2}\varepsilon^2 \int |\nabla \eta|^2\, dx$$
$$+ \varepsilon^2 \int \alpha(6u^2\eta^2 + 4u\eta^3\, \varepsilon + \eta^4\, \varepsilon^2)\, dx$$

($\Rightarrow$) Suppose that u is minimizing. The $u(x) + \varepsilon\eta(x)$ is admissible, and

$$I(u + \varepsilon\eta) \geq I(u)$$

by assumption. We must have $\int(\nabla u \cdot \nabla \eta + 4\alpha u^3 \eta + f(x)\eta) = 0$. (If $I(u) + \varepsilon J + \varepsilon^2(\ldots)$ attains its minimum for $\varepsilon = 0$, then $J = 0$!!!)

($\Leftarrow$) If Euler's equation in weak form holds, then

$$I(u + 1\eta) = I(u) + 0 + \frac{1}{2}\int |\nabla \eta|^2\, dx + \alpha \int \eta^2[u^2 \cdot 6 + 4u\eta + \eta^2]\, dx \geq I(u),$$

since $|\nabla\eta|^2 \geq 0$ and $6u^2 + 4u\eta + \eta^2 \geq 2u^2 \geq 0$. This means that $I(u+\eta) \geq I(u)$, in other words u *is* minimizing. (Any admissible function v can be written as $v = u + (v - u)$, $v - u = \eta \in W_0^{1,2}(\Omega)$.)

[II]

The minimizing function u is unique (if it exists). Suppose that there are two solutions, say u_1 and u_2. Then $u_1 - u_2$ and $u_2 - u_1$ are in $W_0^{1,2}(\Omega)$. Choose the test function $\eta = u_2 - u_1$ in the Euler equation for u_2 and the test function $u_1 - u_2$ in the Euler equation for u_1. Adding the two equations we get (DO IT)

$$\int_\Omega |\nabla u_1 - \nabla u_2|^2\, dx + 4 \int_\Omega \underbrace{\alpha\, \overbrace{(u_1^3 - u_2^3)(u_1 - u_2)}^{\geq (u_1-u_2)^2 \frac{1}{2}(u_1^2+u_2^2)}}_{\geq 0}\, dx = 0.$$

Hence the first integral is zero and so $\nabla u_1 = \nabla u_2$ a.e. in Ω . Thus $u_1 - u_2$ is constant a.e. in Ω (you may wish to prove this in Sobolev's space). The boundary values will force this constant to be zero. This shows that $u_1 = u_2$.

[III]

There exists a minimizing function u. We need some estimates to begin with.

Now

$$\begin{aligned}\left|\int f(x) v\, dx\right| &= \left|\int f(x)(v-\varphi)\, dx + \int f(x)\varphi\, dx\right| \overset{\text{HÖLDER}}{\leq} \|f\|_2 \|v-\varphi\|_2 + \|f\varphi\|_1 \\ &\overset{\text{SOBOLEV}}{\leq} \|f\|_2 C_\Omega \|\nabla(v-\varphi)\|_2 + \|f\varphi\|_1 \\ &\leqq \|f\|_2 C_\Omega \|\nabla v\|_2 + \|f\|_2 C_\Omega \|\nabla \varphi\|_2 + \|f\varphi\|_1 \\ &= A\|\nabla v\|_2 + B\end{aligned}$$

for all v. This means that

$$I(v) \geq \frac{1}{2}\|\nabla v\|_2 \left[\|\nabla v\|_2 - \frac{A}{2}\right] - B$$

and so $I(v) \geq -\frac{A^2}{32} - B$. Hence we have

$$\boxed{-\infty < \operatorname*{Inf}_v I(v) \leq I(\varphi) < +\infty.}$$

Observe also that, for example,

$$\|\nabla v\|_2 \leq \max\{2 + \frac{A}{2},\ B + I(v)\} \approx I(v).$$

By the definition of the infimum, there are admissible functions $u_1, u_2, u_3, \ldots$ such that

$$\lim_{i\to\infty} I(u_i) = I_0 = \inf I(v).$$

This is called a minimizing sequence. We may assume that

$$I_0 \leq I(u_i) < I_0 + 1.$$

By the above bound

$$\boxed{\|\nabla u_i\|_2 \leq \max\left\{2 + \tfrac{A}{2}, 1 + B + I_0\right\} = K}$$

for all $i = 1, 2, 3, \ldots$ Now

$$\|u_i\|_2 \leq \|\varphi\|_2 + \|u_i - \varphi\|_2 \leq \|\varphi\|_2 + C_\Omega \|\nabla u_i - \nabla\varphi\|_2 \leq \|\varphi\|_2 + C_\Omega K + C_\Omega \|\nabla\varphi\|_2$$

for all $i = 1, 2, 3, \ldots$ By the weak compactness of $L^2(\Omega)$ there are functions u and $\bar{w}$ in $L^2(\Omega)$ such that

$$u_{i_\nu} \rightharpoonup u, \quad \nabla u_{i_\nu} \rightharpoonup \bar{w}$$

weakly in $L^2(\Omega)$. We must have that $\bar{w} = \nabla u$ and $u \in W^{1,p}(\Omega)$. Since $u_i - \varphi \in W_0^{1,p}(\Omega)$, so does $u - \varphi$. Now we claim that $I(u) = I_0$, that is, u is the solution.

It is sufficient to establish that

$$I(u) \leq \underline{\lim}_{\nu\to\infty} I(u_\nu)$$

since $I(u) \geq I_0$ (u is admissible!). First,

$$\int_\Omega |\nabla u_{i_\nu}|^2\,dx \geq \int_\Omega |\nabla u|^2\,dx + 2\underbrace{\int_\Omega \nabla u \cdot (\nabla u_{i_\nu} - \nabla u)\,dx}_{\substack{\to 0 \\ \text{by the weak} \\ \text{convergence in } L^2(\Omega)}}$$

and so

$$\underline{\lim}_{\nu\to\infty} \int_\Omega |\nabla u_{i_\nu}|^2\,dx \geq \int_\Omega |\nabla u|^2\,dx.$$

By the Rellich-Kondrachev theorem $u_i - \varphi \to u - \varphi$ *strongly* in $L^2(\Omega)$ at least for some subsequence. Hence, passing again to some subsequence, we have that $u_i(x) \to u(x)$ at a.e. point $x \in \Omega$. Thus

$$\int_\Omega \alpha u^4\;dx \leq \underline{\lim} \int_\Omega \alpha u_i^4\,dx$$

by Fatous Lemma, at least for a subsequence. Finally,

$$\int f(x)u_{i_\nu}\,dx \to \int f(x)u\,dx$$

by weak convergence. Collecting results, we have the desired semicontinuity. □

1) Is u continuous? What about $u \in C^2(\Omega)$, that is, is u a classical solution? This is REGULARITY THEORY (de Giorgi, Moser, Nash).

2) Are the boundary values attained in the classical sense; $\lim\limits_{\substack{x\to\xi \\ x\in\Omega}} u(x) = \varphi(x),\ \xi \in \partial\Omega$? (This is true only in "regular domains.")

3) Stability? What do small changes of the data cause?

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