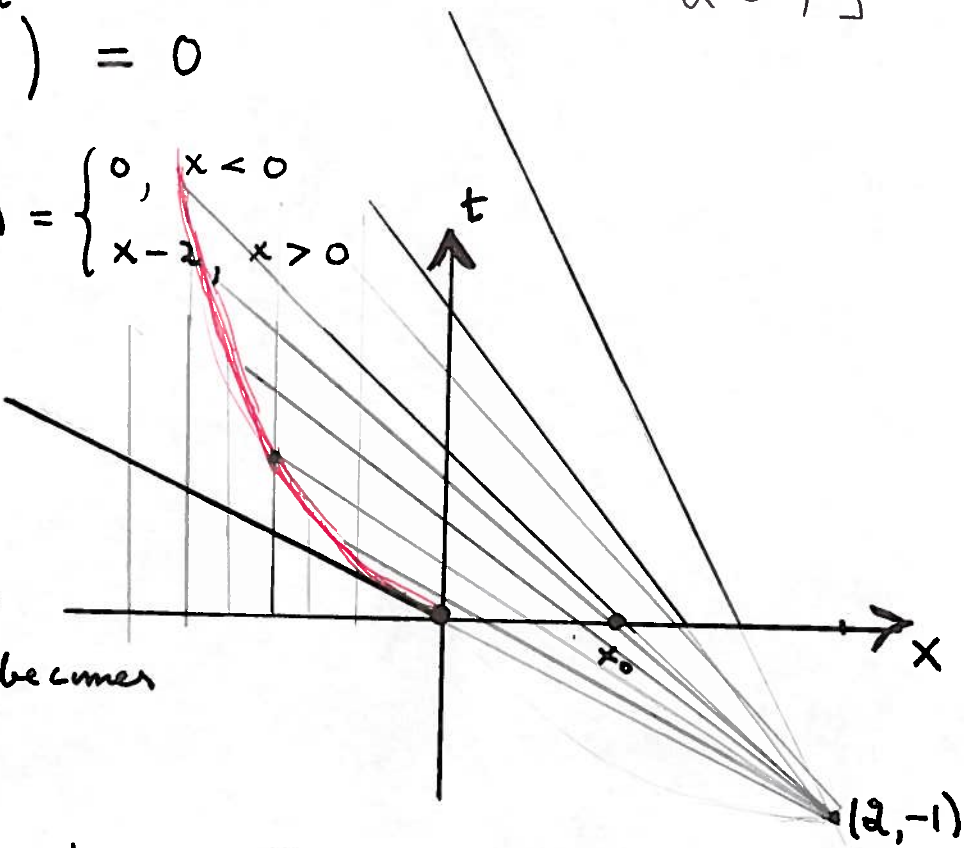
2015

(1)
$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{u^2}{2}\right) = 0 \\[2ex] u(x,0) = g(x) = \begin{cases} 0, & x<0 \\ x-2, & x>0 \end{cases} \end{cases}$$



A shock curve emerges at the origin. To the right of it the solution becomes

$$u = u_{right} = \frac{x-2}{t+1},$$

$$\begin{cases} u(x,t) = x_0 - 2 \\ \text{along the line} \\ x - x_0 = -t(2-x_0) \end{cases}$$

to the left

$$u = u_{left} = 0$$

The Rankine – Hugoniot shock condition reads

$$\frac{dx}{dt}\left[\frac{x-2}{t+1} - 0\right] = \frac{1}{2}\left[\left(\frac{x-2}{t+1}\right)^2 - 0^2\right]$$

$$\frac{dx}{dt} = \frac{1}{2}\,\frac{x-2}{t+1}$$

Upon integration

$$t+1 = C(x-2)^2, \quad C = \frac{1}{4} \text{ (the origin!)}$$

SHOCK : $$\boxed{t = -1 + \frac{1}{4}(x-2)^2}$$ Parabola.

(2) Use the solid mean value property for harmonic functions

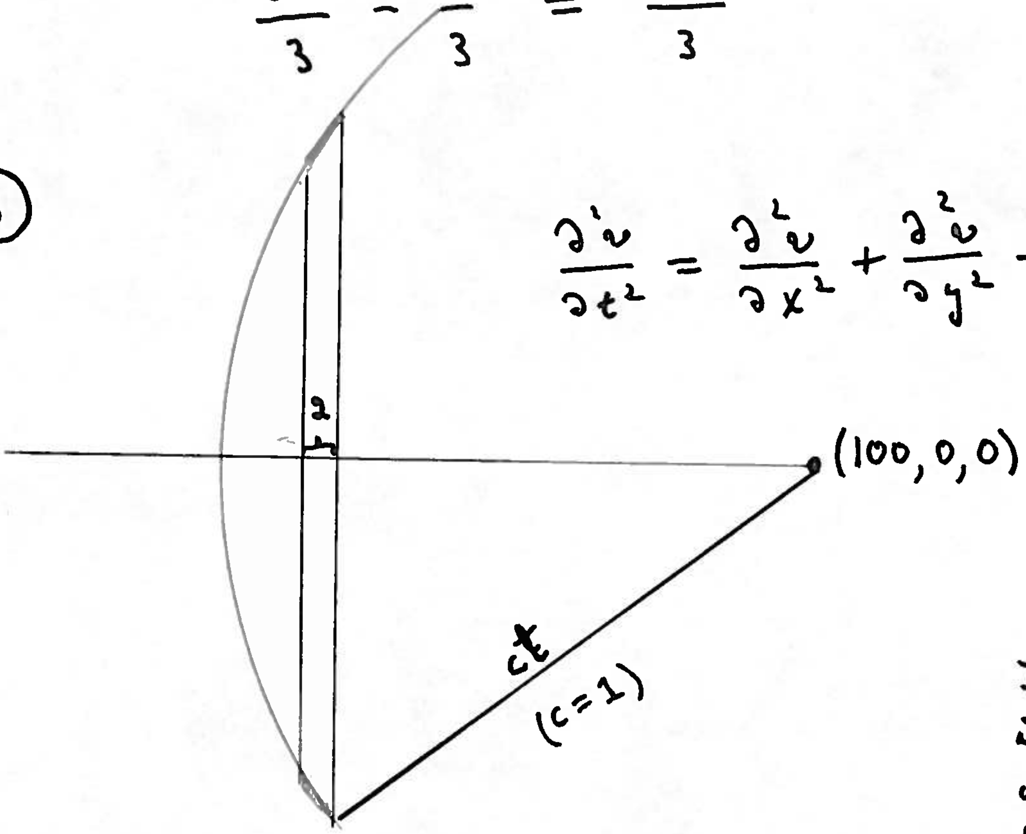
$$h(0,0,0) = \frac{1}{\frac{4}{3}\pi \cdot 2^3} \iiint_{x^2+y^2+z^2<2^2} h \, dx\,dy\,dz = \frac{1}{\frac{4}{3}\pi \cdot 1^3} \cdot \underbrace{\frac{\pi}{3}}_{\text{given}}$$

to obtain the volume

$$\frac{8\pi}{3} - \frac{\pi}{3} = \frac{7\pi}{3}$$

(3)

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}$$



The integration is over the part of the sphere wich lies between the two planes $x = \pm 1$.

Kirchhoff's formula yields

$$v(100,0,0,t) = \frac{t}{4\pi t^2} \iint_{\substack{S(100,0,0;\,t) \\ |\xi| \le 1}} 13 \, dS_t$$

$$= \frac{t}{4\pi t^2} \underbrace{2\pi t \cdot 2 \cdot 13}_{\text{area of the zone } |x|<1 \text{ on the sphere}} = 13.$$ Also the limit is 13 as $t \to \infty$.

④ The initial condition $v(x,0) = 0$ implies that $v_x(x,0) = 0$. In the same way, $v_t(0,t) = 0$, $v_t(1,t) = 0$.

$$\mathcal{E}(t) = \frac{1}{2}\int_0^1 \left(v_t^2 + c^2 v_x^2 + e^x v^2\right) dx$$

$$\dot{\mathcal{E}}(t) = \int_0^1 \left(v_t v_{tt} + c^2 v_x v_{xt} + e^x v v_t\right) dx$$

$$= \int_0^1 v_t \underbrace{\left(v_{tt} - c^2 v_{xx} + e^x v\right)}_{\equiv 0 \text{ by the diff. eqn}} dx \qquad = 0$$

since

$$\int_0^1 v_x v_{xt}\, dx = \underbrace{\Big[v_x v_t \Big]_0^1}_{=0 \quad v_t(1,t)=0,\ v_t(0,t)=0} - \int_0^1 v_t v_{xx}\, dx .$$

Thus $\mathcal{E}(t) \equiv$ Constant. $\mathcal{E}(0) = 0.$ *)
Hence $\mathcal{E}(t) = 0$. But all the terms in the integral $\mathcal{E}(t)$ are ≥ 0. Hence they must be zero to yield $\mathcal{E}(t) = 0$. It follows that $v \equiv 0$.

*) $$\mathcal{E}(0) = \frac{1}{2}\int_0^1 \Big(\underbrace{v_t(x,0)}_{=0}{}^2 + c^2 \underbrace{v_x(x,0)}_{=0}{}^2 + e^x \underbrace{v(x,0)}_{=0}{}^2\Big) dx \qquad = 0$$

⑤ Let $\eta = \eta(x,y,z)$ be a smooth function with boundary values 0 on the sides of the cube. Assume that u is a minimizer. Consider the competing function $u(x,y,z) + \varepsilon\eta(x,y,z)$. It is necessary for a minimum that

$$\boxed{\frac{d}{d\varepsilon} I(u+\eta\varepsilon) = 0, \text{ when } \varepsilon = 0.}$$

This yields

$$\int_0^1\int_0^1\int_0^1 \left(2e^x u_x \eta_x + 2e^y u_y \eta_y + 2e^z u_z \eta_z - e^{xyz}\eta\right) dx\,dy\,dz = 0$$

Integrations by parts give

$$-\int_0^1\int_0^1\int_0^1 \left[2\frac{\partial}{\partial x} e^x u_x + 2\frac{\partial}{\partial y} e^y u_y + 2\frac{\partial}{\partial z} e^z u_z + e^{xyz}\right]\eta\, dx\,dy\,dz$$

$$= 0$$

Since this must hold for <u>all</u> such η, the [ ]-term must be zero (Variational Lemma, du Bois-Reymond.) The <u>Euler</u> – <u>Lagrange</u> equation is

$$\frac{\partial}{\partial x}\left(e^x \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(e^y \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(e^z \frac{\partial u}{\partial z}\right) + \frac{1}{2} e^{xyz} = 0,$$

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which can be "expanded".

(6) The simplest solution seems to be the following. If $w \in C^2(\bar{\Omega})$ has a maximum at an interior point $x_0 \in \Omega$, then

$$(1) \qquad \Delta w(x_0) \leq 0$$

by the infinitesimal calculus. If $w(x_0) \geq 11$, then

$$(2) \quad \Delta w(x_0) \geq 11(11-10) - 10 = 1$$

Now (1) and (2) are incompatible. Thus $w(x_0) < 11$ at interior maximum points x_0. If the maximum is attained at the boundary $\partial\Omega$, then $w \leq 5$.

In all cases $w < 11$ in $\bar{\Omega}$.

<u>Remark</u> At an interior maximum

$$\frac{\partial w}{\partial x_1} = 0, \quad \frac{\partial w}{\partial x_2} = 0, \quad \dots, \quad \frac{\partial w}{\partial x_n} = 0$$

$$\frac{\partial^2 w}{\partial x_1^2} \leq 0, \quad \frac{\partial^2 w}{\partial x_2^2} \leq 0, \quad \dots, \quad \frac{\partial^2 w}{\partial x_n} \leq 0$$

for functions with continuous second derivatives.

⑥ ALTERNATIVE WAY. (appendix)

$$\Delta w = w(w-10) - 10$$

Let us first, for simplicity, show that $w \leq 11$ (instead of $w < 11$). The open set

$$\Omega_{11} = \{ x \in \Omega \mid w(x) > 11 \}$$

cannot touch the boundary $\partial\Omega$ of Ω, since $w = 5 < 11$ on $\partial\Omega$. Now

$$\Delta w = w(w-10) - 10 > 11 \cdot (11-10) - 10$$
$$= 1 \quad \text{in } \Omega_{11}$$

Thus w cannot have an interior maximum in Ω (at max. points $\Delta w \leq 0$ by calculus). Therefore

$$w \leq \max_{\partial\Omega_{11}} w = 11 \quad \text{in } \Omega_{11}$$

Since $w \leq 11$ in Ω_{11}, Ω_{11} was empty. We have proved that $w \leq 11$ in Ω.

To get $w < 11$ (strict) one has to replace 11 by $10{,}99$ in the above reasoning. This yields $w \leq 10{,}99 < 11$. (Also $10{,}92$ will do but not $10{,}9$.)