NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for
TMA4305 Partial Differential Equations

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Permitted examination support material: Ett gult A4-ark stemplet fra Instituttet med valgfri paaskrift av studenten. Bestemt, enkel kalkulator tillaten.

Other information:
There are 6 problems.

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Problem 1 Consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = 0 \\ u(x,0) = g(x). \end{cases}$$

Find the solution for

$$g(x) = \begin{cases} 0, & \text{when} \quad x < 0 \\ x - 2, & \text{when} \quad x > 0. \end{cases}$$

In particular, determine the shock curve in the xt-plane starting at the origin.

Problem 2 The function $h = h(x, y, z)$ is harmonic in the whole space $\mathbb{R}^3$, i. e., $\Delta h = 0$. Given that

$$\iiint\limits_{x^2+y^2+z^2<1} h(x,y,z)\,dx\,dy\,dz = \frac{\pi}{3},$$

find

$$\iiint\limits_{1<x^2+y^2+z^2<4} h(x,y,z)\,dx\,dy\,dz = ?$$

where the latter integration is over the domain $1 < x^2 + y^2 + z^2 < 4$.

Problem 3 Let $v = v(x, y, z, t)$ be the solution of the wave equation

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}$$

that has the initial values

$$v(x,y,z,0) = 0,\; v_t(x,y,z,0) = \begin{cases} 13 & \text{if} \quad |x| < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Determine $v(100, 0, 0, t)$ for $t > 101$. Find the limit $\lim v(100, 0, 0, t)$ as $t \to \infty$.[1]

[1] The area of a zone on a sphere has the formula "$2\pi rh$" and the sphere has area $4\pi r^2$ (Archimedes).

Problem 4 The problem

$$\begin{cases} v_{tt} = c^2 v_{xx} - e^x v & (0 < x < 1,\ t > 0) \\ v(x,0) = 0,\ v_t(x,0) = 0,\ v(0,t) = 0,\ v(1,t) = 0 \end{cases}$$

has at most one solution $v = v(x,t)$ with continuous second derivatives in the region $0 \le x \le 1$, $t \ge 0$. Prove this uniqueness using the "energy"

$$E(t) = \frac{1}{2}\int_0^1 \Big(v_t(x,t)^2 + c^2 v_x(x,t)^2 + e^x v(x,t)^2\Big)\,dx.$$

Problem 5 Assume that the variational integral

$$I(u) = \int_0^1\int_0^1\int_0^1 \Big(e^x u_x^2 + e^y u_y^2 + e^z u_z^2 - e^{xyz} u\Big)\,dx\,dy\,dz$$

has a minimum among all sufficiently smooth functions $u = u(x,y,z)$ with boundary values 0 on the sides of the cube. Find a second order differential equation for the minimizer (the Euler-Lagrange Equation).

Problem 6 Suppose[2] that $w \in C^2(\overline{\Omega})$ is a solution to the equation

$$\Delta w = w(w - 10) - 10$$

with boundary values $w = 5$ on $\partial\Omega$. Here Ω is a bounded domain in space. Show that $w < 11$ in Ω.

Good luck!

[2]The notation means that the function has second derivatives that are continuous up to the boundary of the domain