ELLIPTIC 2nd ORDER EQS

$$L(u) \equiv \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x)\frac{\partial u}{\partial x_j} + \cancel{C(x)u} = f(x)$$

$(x = (x_1, x_2, \dots, x_n) \in \Omega)$

DRIFT or CONVECTION TERM

Keep $c=0$! for simplicity.

We assume

Can always be arranged!

1) $a_{ij} = a_{ji}$ (symmetry)

2) $\displaystyle\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \gamma|\xi|^2, \quad \gamma > 0,$

It follows that $a_{11}(x) \geq \gamma$.

for all $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ (uniform ELLIPTICITY)

3) bounded coefficients (needed $b_1(x) \geq -M$)

The following problems appear:

- Existence of a solution with given boundary data (Dirichlet or Neumann cond.)
- Uniqueness
- Stability
- Maximum/minimum Principles
- Regularity of solutions (e.g. differentiability)
- Boundary behaviour
- Etc.

Trace$(B) = b_{11} + b_{22} + \cdots + b_{nn}$
$=$ sum of diagonal elements.

In matrix notation

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = \text{Trace}(A\mathbb{D})$$

where $A = (a_{ij})$, $\mathbb{D} = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)$ are symmetric $n \times n$ - matrices: $A = A^T$, $\mathbb{D} = \mathbb{D}^T$ (transposes).

LEMMA At an interior maximum point for $u = u(x)$ we have $L(u) \leq 0$; at an interior minimum $L(u) \geq 0$.

Proof for minimum. By assumption

$$\langle A\xi, \xi\rangle \equiv \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \geq 0,$$

abbreviated $A \geq 0$, and by the infinitesimal calculus $\mathbb{D} \geq 0$. At the minimum $\nabla u = \bar{0}$. Thus

$$Lu = \text{Trace}(A\mathbb{D}).$$

Diagonalize A (possible for symmetric matrices):

$$\Lambda = SAS^T, \quad A = S^T \Lambda S$$

$$\Lambda = (\lambda_j \delta_{ji}), \quad \lambda_j > 0 \qquad \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \lambda_n \end{pmatrix}$$

$$\Lambda^{1/2} = (\sqrt{\lambda_j}\, \delta_{ji}) \quad \text{square root}$$

$\lambda_i \geq 0$ since $0 \leq \langle A\xi, \xi\rangle = \langle S^T\Lambda S\xi, \xi\rangle = \langle \Lambda \underbrace{S\xi}_{\eta}, \underbrace{S\xi}_{\eta}\rangle = \langle \Lambda\eta, \eta\rangle = \sum \lambda_i \eta_i^2$

We need the rule $\text{Trace}(BC) = \text{Trace}(CB)$.

Thus, at the minimum,

$$\Lambda^{\frac{1}{2}} = \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots \\ & & & \sqrt{\lambda_n} \end{pmatrix}.$$

$$\begin{aligned} \underline{L(u)} &= \mathrm{Trace}\,(S^T \Lambda S \mathbb{D}) \\ &= \mathrm{Trace}\,(\underbrace{S^T \Lambda^{1/2}}_{\mathrm{I}} \underbrace{\Lambda^{1/2} S \mathbb{D}}_{\mathrm{II}}) \\ &= \mathrm{Trace}\,(\underbrace{\Lambda^{1/2} S \mathbb{D}}_{\mathrm{II}} \underbrace{S^T \Lambda^{1/2}}_{\mathrm{I}}) \\ &= \mathrm{Trace}\,((\Lambda^{1/2} S) \mathbb{D} (\Lambda^{1/2} S)^T) \qquad B = \Lambda^{\frac{1}{2}} S \\ &= \mathrm{Trace}\,(B \mathbb{D} B^T) \\ &\underline{\geq 0}, \end{aligned}$$

because $\mathbb{D} \geq 0$. Indeed,

Transposed.

$$(B\mathbb{D})_{ik} = \sum_{\ell} b_{i\ell} d_{\ell k}, \quad (B \mathbb{D} B^T)_{ij} = \sum_k \Big(\sum_\ell b_{i\ell} d_{\ell k} \Big) b_{jk}$$

$$\begin{aligned} (B \mathbb{D} B^T)_{jj} &= \sum_{k=1}^{n} \sum_{\ell=1}^{n} b_{j\ell} d_{\ell k} b_{jk} \\ &= \langle \mathbb{D} b^j, b^j \rangle \geq 0, \quad b^j = (b_{j1}, b_{j2}, \dots, b_{jn}) \text{ "}= \xi\text{"} \end{aligned}$$

$$\mathrm{Trace}\,(B\mathbb{D}B^T) = \sum_{j=1}^{n} \langle \mathbb{D} b^j, b^j \rangle \geq 0. \quad \blacksquare \qquad \text{QED.}$$

closure

PROPOSITION Assume that $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ where Ω is a bounded domain. If $L(u) \equiv 0$ then

$$\min_{\partial\Omega} u \leq u \leq \max_{\partial\Omega} u.$$

<u>Proof</u> for minimum. At an interior minimum for the auxiliary function

$$v_\varepsilon(x) = u(x) + \varepsilon e^{\alpha x_1}$$

we have

LEMMA

$$0 \le L(v_\varepsilon) = L(u) - \varepsilon a_{11}\alpha^2 e^{\alpha x_1} - b_1 \varepsilon \alpha e^{\alpha x_1}$$

$$= -\varepsilon\alpha e^{\alpha x_1}(\alpha a_{11} + b_1)$$

$$\le -\varepsilon\alpha e^{\alpha x_1}(\alpha\gamma - \inf b_1)$$

$$\le -\varepsilon\alpha e^{\alpha \min_\Omega x_1}[\alpha\gamma - \inf b_1] < 0 \quad \text{for} \quad \alpha > \frac{\inf b_1}{\gamma}.$$

The rôle of α.

This is a contradiction. It follows that v_ε cannot have interior minima:

$$u(x) - \varepsilon e^{\alpha x_1} \ge \min_{x\in\partial\Omega}\left(u(x) - \varepsilon e^{\alpha x_1}\right)$$

$$\ge \min_{x\in\partial\Omega} u(x) - \varepsilon \max_{\partial\Omega} e^{\alpha x_1}$$

Let $\varepsilon \to 0$. Thus

$$u(x) \ge \min_{\partial\Omega} u.$$

Ω IS A BOUNDED DOMAIN

THEOREM The solution $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ to the equation $L(u) = f$ with boundary values $u = g$ on $\partial\Omega$ is unique (if it exists).

<u>Proof</u>: If there are two solutions, say u_1, u_2, then $w = u_2 - u_1$ solves the problem

$$L(w) = 0, \quad w = 0 \text{ on } \partial\Omega.$$

By the Proposition

$$0 = \min_{\partial\Omega} w \le w \le \max_{\partial\Omega} w = 0$$

$$\Longrightarrow w = 0 \text{ in } \Omega.$$

Q.E.D.