Peter Lindqvist 2014.

VANISHING VISCOSITY

The solution of

$$\begin{cases} u_t + u u_x = \varepsilon u_{xx} \\ u(x,0) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases} \end{cases} \qquad (u = u_\varepsilon)$$

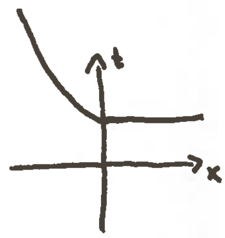
should approach the shock solution (shock curve $t = 2x$) as $\varepsilon \to 0+$. Limit eqn. $\underline{u_t + u u_x = 0}$

Cole-Hopf transf.

$$\underbrace{\frac{\partial}{\partial t} u}_{-\psi_x} + \frac{\partial}{\partial x} \underbrace{\left(\frac{u^2}{2} - \varepsilon u_x \right)}_{\psi_t} = 0$$

$$\psi_x = -u, \quad \psi_t = \frac{1}{2} \psi_x^2 + \varepsilon \psi_{xx}$$

$$2\varepsilon \ln \phi = \psi, \quad \boxed{\phi_t = \varepsilon \phi_{xx}}$$ HEAT EQN.



$$\boxed{u = -2\varepsilon \frac{\partial}{\partial x} \ln \phi}$$

Initial values: $\phi(x,0) = \begin{cases} e^{-x/2\varepsilon}, & x < 0 \\ 1, & x \geq 0 \end{cases}$

(The solution $\phi = \phi(x,t)$ is unique among the non-negative ones.)

$$\phi(x,t) = \frac{1}{\sqrt{4\pi\varepsilon t}} \int_{-\infty}^{0} e^{-\frac{y}{2\varepsilon}} e^{-\frac{(x-y)^2}{4\varepsilon t}} \, dy$$

$$+ \frac{1}{\sqrt{4\pi\varepsilon t}} \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4\varepsilon t}} \, dy$$

$$\frac{\partial \phi(x,t)}{\partial x} \overset{*)}{=} \frac{1}{\sqrt{4\pi\varepsilon t}} \int_{-\infty}^{0} e^{-\frac{y}{2\varepsilon}} \frac{\partial}{\partial y}\left(-e^{-\frac{(x-y)^2}{4\varepsilon t}}\right) dy$$

*) $\frac{\partial}{\partial x} e^{-\frac{(x-y)^2}{4\varepsilon t}} = -\frac{\partial}{\partial y} e^{-\frac{(x-y)^2}{4\varepsilon t}}$

$$\frac{1}{\sqrt{4\pi\varepsilon t}} \int_{0}^{\infty} \frac{\partial}{\partial y}\left(-e^{\frac{(x-y)^2}{4\varepsilon t}}\right) dy$$

$$= -\frac{1}{\sqrt{4\pi\varepsilon t}} \Big/_{-\infty}^{0} e^{-y/2\varepsilon} e^{-\frac{(x-y)^2}{4\varepsilon t}} - \frac{1}{\sqrt{4\pi\varepsilon t}} \int_{-\infty}^{0} \frac{1}{2\varepsilon} e^{-\frac{y}{2\varepsilon}} e^{-\frac{(x-y)^2}{4\varepsilon t}} dy$$

$$- \frac{1}{\sqrt{4\pi\varepsilon t}} \Big/_{0}^{\infty} e^{-\frac{(x-y)^2}{4\varepsilon t}}$$

$$= \cancel{-\frac{1}{\sqrt{4\pi\varepsilon t}} e^{-x^2/4\varepsilon t}} + \cancel{\frac{1}{\sqrt{4\pi\varepsilon t}} e^{-x^2/4\varepsilon t}} - \frac{1}{2\varepsilon} \int_{-\infty}^{0} e^{-y/2\varepsilon} e^{-\frac{(x-y)^2}{4\varepsilon t}} dy \cdot \frac{1}{\sqrt{4\pi\varepsilon t}}$$

We write u for u_ε below.

$$u(x,t) = -2\varepsilon \frac{\phi_x(x,t)}{\phi(x,t)}$$

$$= \frac{\frac{1}{\sqrt{4\pi\varepsilon t}} \int_{-\infty}^{0} e^{-y/2\varepsilon} e^{-\frac{(x-y)^2}{4\varepsilon t}} dy}{\frac{1}{\sqrt{4\pi\varepsilon t}} \int_{-\infty}^{0} e^{-y/2\varepsilon} e^{-\frac{(x-y)^2}{4\varepsilon t}} dy + \frac{1}{\sqrt{4\pi\varepsilon t}} \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4\varepsilon t}} dy}$$

$$\frac{y}{2\varepsilon} + \frac{(x-y)^2}{4\varepsilon t} = \frac{2ty + x^2 - 2xy + y^2}{4\varepsilon t}$$

$$= \frac{x^2 - 2y(x-t) + y^2}{4\varepsilon t} = \frac{(x-t-y)^2 - t^2 + 2xt}{4\varepsilon t}$$

by completing the square!

$e^{-y/2\varepsilon - \frac{(x-y)^2}{4\varepsilon t}} = \ldots$

$$\frac{1}{\sqrt{4\pi\varepsilon t}}\int_{-\infty}^{0} e^{-y/2\varepsilon}\, e^{-\frac{(x-y)^2}{4\varepsilon t}}\,dy \qquad\qquad \frac{x-t-y}{\sqrt{4\varepsilon t}} = -s$$

$$= \frac{1}{\sqrt{4\pi\varepsilon t}}\, e^{\frac{t^2-2xt}{4\varepsilon t}} \int_{-\infty}^{0} e^{-\frac{(x-t-y)^2}{4\varepsilon t}}\,dy$$

$$= \frac{1}{\sqrt{\pi}}\, e^{\frac{t^2-2xt}{4\varepsilon t}} \int_{-\infty}^{(t-x)/\sqrt{4\varepsilon t}} e^{-s^2}\,ds \quad \left(\xrightarrow[\varepsilon\to 0+]{} \infty \text{ if } t>2x>0\right)$$

$$\frac{1}{\sqrt{4\pi\varepsilon t}}\int_{0}^{\infty} e^{-\frac{(x-y)^2}{4\varepsilon t}}\,dy = \frac{1}{\sqrt{\pi}}\int_{-\frac{x}{\sqrt{4\varepsilon t}}}^{\infty} e^{-s^2}\,ds \xrightarrow[\varepsilon\to 0+]{} \begin{cases} 0, & x<0 \\ 1, & x>0 \end{cases}$$

$$M(x,t) = \frac{\dfrac{1}{\sqrt{\pi}}\displaystyle\int_{-\infty}^{\frac{t-x}{\sqrt{4\varepsilon t}}} e^{-s^2}\,ds}{\dfrac{1}{\sqrt{\pi}}\displaystyle\int_{-\infty}^{(t-x)/\sqrt{4\varepsilon t}} e^{-s^2}\,ds + e^{-\frac{t(t-2x)}{4\varepsilon t}}\,\dfrac{1}{\sqrt{\pi}}\displaystyle\int_{-x/\sqrt{4\varepsilon t}}^{\infty} e^{-s^2}\,ds}$$

$$\xrightarrow[\varepsilon\to 0+]{} \begin{cases} 1, & x<0 \\ 1, & t>2x,\ x>0 \\ 0, & t<2x \end{cases}$$