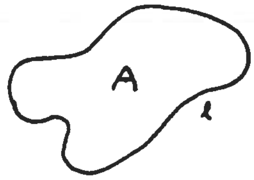
CALCULUS OF VARIATIONS

PETER LINDQVIST
DEPT OF MATHS, NORWEGIAN
INSTITUTE OF TECHNOLOGY
N-7034 NORWAY

One of the oldest problems is <u>the isoperimetric inequality</u>

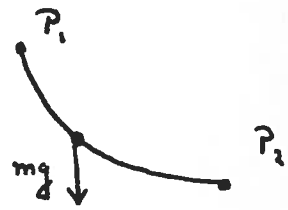
$$A \leq \frac{1}{4\pi} \ell^2$$



A = area
ℓ = length of the perimeter.

for the area of a plane figure with perimeter ℓ. The optimal figure is a circle. A more elaborate variant is the problem of Queen Dido. — In 1696 J. Bernoulli proposed the celebrated <u>brachystochrone problem</u> that leads to the minimization of the integral

$$T = \int_{x_1}^{x_2} \sqrt{\frac{1 + y'^2(x)}{2g(y_0 - y(x))}}\, dx$$



among all curves $y = y(x)$ joining the points (x_1, y_1) and (x_2, y_2). The solution is a <u>cycloid</u>.

For functions of several variables, the most celebrated examples are

DIRICHLET INT.

$$D(u) = \int_\Omega |\nabla u|^2 dx, \qquad A(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx,$$

AREA INT.

$$R(u) = \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}$$

RAYLEIGH QUOTIENT

where $\Omega \subset \mathbb{R}^n$ is a bounded domain. The problem of minimizing the Dirichlet integral $D(u)$ among all functions with given boundary values leads to the Laplace equation $\Delta u = 0$ (neither the Heat Equation nor the Wave Equation can be obtained from a variational principle). Minimizing the Rayleigh Quotient $R(u)$ among all functions u with boundary values zero on $\partial\Omega$ yields us the first eigenvalue of the equation $\Delta u + \lambda u = 0$.

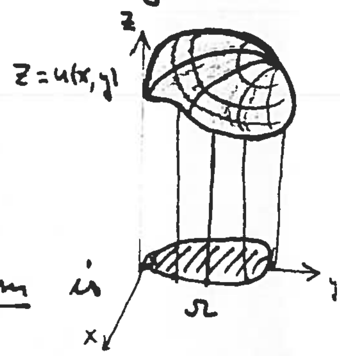
Formally, the problem of minimizing the area integral $A(u)$ among all functions u with given boundary values φ on the boundary of the bounded domain $\Omega \subset \mathbb{R}^2$, leads to the <u>minimal surface equation</u>

$$(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy} = 0.$$

To see this, suppose that u is minimizing and choose any test-function $\eta \in C_0^\infty(\Omega)$. Then $u(x,y) + \varepsilon\eta(x,y)$ has the right boundary values, if u has, so that

$$A(u) \leq A(u+\varepsilon\eta), \qquad (-\infty < \varepsilon < \infty)$$

that is, the function $J(\varepsilon) = A(u+\varepsilon\eta)$ attains its minimum for $\varepsilon = 0$. Then $\underline{J'(0) = 0}$. Calculating, we get



$$J'(\varepsilon) = \iint_\Omega \frac{(\nabla u + \varepsilon\nabla\eta)\cdot\nabla\eta}{\sqrt{1+|\nabla(u+\varepsilon\eta)|^2}}\,dx\,dy,$$

so that <u>the necessary condition for a minimum</u> is

$$\iint_\Omega \frac{\nabla u\cdot\nabla\eta}{\sqrt{1+|\nabla u|^2}}\,dx\,dy = 0.$$

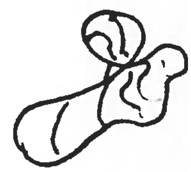
EULER-LAGRANGE EQUATION (IN WEAK FORM)

If we know that $u \in C^2(\Omega)$, then an integration by parts yields

$$\iint\limits_{\Omega} \eta\left[\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{}}\right)+\frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{}}\right)\right]dx\,dy = 0,$$

whenever $\eta \in C_0^\infty(\Omega)$. By the <u>Variational Lemma</u>, the expression $[\;] = 0$. Simplifying, we arrive at the equation for minimal surfaces.

<u>Remark:</u> 1) This was a special case of PLATEAU's problem; the general case leads to the study of <u>parametric integrals</u>



$$\iint \sqrt{\left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2+\left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2+\left(\frac{\partial(z,x)}{\partial(s,t)}\right)^2}\;ds\,dt$$

The existence of a solution was established in the 1930's by T. Radó and J. Douglas.

2) The minimal surface equation has solutions that are not area minimizing. Also such "false solutions" are said to represent <u>minimal surfaces</u>. (Uniqueness fails.)

To derive the Euler (or the Euler-Lagrange) Equation for the variational integral

$$I(u) = \int\limits_{\Omega} F(x, u(x), \nabla u(x))\,dx,$$

one proceeds formally as above; the minimization is over a suitable class of functions, for example $W^{1,p}(\Omega)$, all the admissible functions having the same boundary values. Choose $\eta \in C_0^\infty(\Omega)$. Then

$$I(u) \le I(u+\varepsilon\eta) \qquad (-\infty < \varepsilon < \infty)$$

if u is minimizing. The necessary condition for a minimum is that

$$\left[\frac{d\,I(u+\varepsilon\eta)}{d\varepsilon}\right]_{\varepsilon=0} = 0.$$

Writing $F = F(x,u,w)$, when $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, and $w \in \mathbb{R}^n$, and using the abbreviations F_u, $\nabla_w F = \left(\frac{\partial F}{\partial w_1}, \ldots, \frac{\partial F}{\partial w_2}\right)$, we have

EULER–LAGRANGE

$$\int_\Omega [F_u(x,u,\nabla u)\eta + \nabla_w F(x,u,\nabla u)\cdot\nabla\eta]\,dx = 0.$$

This is the Euler–Lagrange equation (in weak form). Of course, appropriate assumptions about F must be made in order that the previous calculation be valid. The Euler–Lagrange equation now takes the form

$$\underbrace{\sum \frac{\partial}{\partial x_i} F_{w_i}(x,u,\nabla u)}_{F_{w_ix_i} + F_{w_iu}\,u_{x_i} + F_{w_iw_j}\,u_{x_jx_i}} = F_u(x,u,\nabla u)$$

SUM OVER REPEATED INDICES

provided that all the derivatives do exist.

The DIRECT METHOD in the Calculus of Variations can be used to prove the existence of minimizers to certain integrals

$$I(u) = \int_\Omega F(x,u(x),\nabla u(x))\,dx.$$

Here Ω is a bounded domain in $\mathbb{R}^n$, $\varphi \in W^{1,p}(\Omega)$ is a given function representing the boundary values.

Any $u \in W^{1,p}(\Omega)$ with $u - \varphi \in \overset{\circ}{W}{}^{1,p}(\Omega)$ is admissible. Usually $p = 2$, but $p > 1$ is allowed. The procedure is as follows

STEP I Show that $-\infty < \inf_v I(v) < +\infty$

STEP II Choose a minimizing sequence $u_1, u_2, \ldots$, that is, $I(u_i) \longrightarrow I_0 = \inf I(v)$.

STEP III Show that $\|u_i\|_{1,p,\Omega} \leq M < \infty$, $i = 1, 2, 3, \ldots$ for some $p > 1$. Hence

$u_{i_\nu} \longrightarrow u$, $\nabla u_{i_\nu} \longrightarrow \nabla u$ weakly in $L^p(\Omega)$ for some admissible u. (This is the weak compactness.)

STEP IV Show that

$$I(u) \leq \lim_{\nu \to \infty} I(u_{i_\nu}).$$

(This property cannot hold for all weakly convergent sequences, unless the mapping $w \longmapsto F(x, u, w)$ is convex!)

Then $I(u) \leq I_0$ and, since u is admissible, $I(u) \geq I_0$. Hence u is the minimizer sought for.

The direct method works for the Dirichlet integral. It does not work for the area integral $A(u)$, because $\sqrt{1 + |\nabla u|^2} \approx |\nabla u|^1$, that is, $p = 1$. (However, one can modify the method to cover this case, though not in any easy way.)