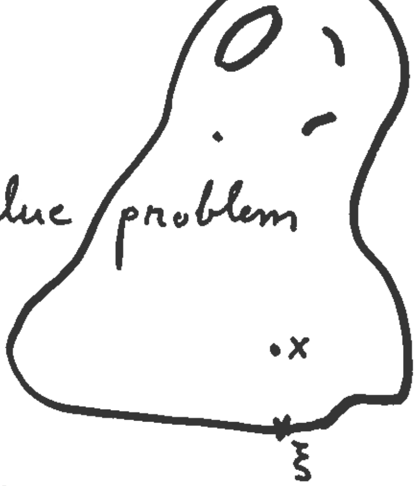
Peter Lindqvist
2004

THE PERRON METHOD

1922



Consider the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$

for a <u>bounded</u> domain Ω in $\mathbb{R}^n$. Here $f : \partial\Omega \longrightarrow [-\infty, \infty]$ is a given function defined on the boundary. The celebrated PERRON method produces the <u>upper solution</u> $\overline{H}_f$ and the <u>lower solution</u> $\underline{H}_f$. Always

$$-\infty \leqq \underline{H}_f \leqq \overline{H}_f \leqq +\infty.$$

They are harmonic functions or identically $+\infty$ or identically $-\infty$. If f is continuous, then $\overline{H}_f = \underline{H}_f$ = a harmonic function (N. Wiener's 1924 Resolutivity theorem). Although coinciding, they can take wrong boundary values at irregular boundary points! In such a case there is no solution with the prescribed boundary values at each point. The concept of a <u>barrier</u> is related to the question

$$\lim_{x \to \xi} H_f(x) \overset{?}{=} f(\xi).$$

The main ingredients are the superharmonic and the subharmonic functions.

HARMONIC FUNCTIONS

In the ball $B(0, R)$ the POISSON INTEGRAL

$$u(x) = \frac{R^2 - |x|^2}{\omega_n R} \oint_{|y| = R} \frac{f(y)}{|x-y|^n}\, dS(y) \qquad (|x| < R)$$

POISSON

represents a function that is harmonic and

$$\lim_{x \to \xi} u(x) = f(\xi), \quad |\xi| = R$$

provided that f is continuous. In particular

$$u(0) = \frac{1}{\omega_n R^{n-1}} \oint_{|y|=R} f(y)\, dS(y),$$

which is the Mean Value taken over the sphere. If $u \geq 0$ and $\Delta u = 0$, then the

HARNACK INEQUALITY

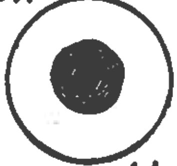
$$\frac{R - |x|}{(R + |x|)^{n-1}} R^{n-2} u(0) \leq u(x) \leq \frac{R + |x|}{(R - |x|)^{n-1}} R^{n-2} u(0)$$

holds, when $|x| < R$. (<u>Proof</u>: $R - |x| < |x - y| < R + |x|$, when $|y| = R$. Use Poisson's integral.)

In particular, if $u \geq 0$ and $\Delta u = 0$ in $B(0,R)$, then

$$\frac{2^{n-2}}{3^{n-1}} u(0) \leq u(x) \leq 3 \cdot 2^{n-2} u(0), \quad \text{when } |x| \leqq \frac{R}{2}$$

Thus we have a <u>uniform</u> <u>bound</u> in the smaller ball.



Suppose that u is harmonic in the domain Ω. Poisson's integral for the ball $\overline{B(x_0, R)} \subset \Omega$ yields

$$u(x) = \frac{R^2 - |x-x_0|^2}{\omega_n R} \oint_{|y-x_0|=R} \frac{u(y)\, dS(y)}{|x-y|^n}$$

when $|x-x_0| < R <$ distance$(x_0, \partial\Omega)$. From this representation several properties follow:

1°) $u \in C^\infty(\Omega)$. It is even <u>real-analytic</u>.

2°) The limit of a uniformly convergent sequence of harmonic functions is itself harmonic

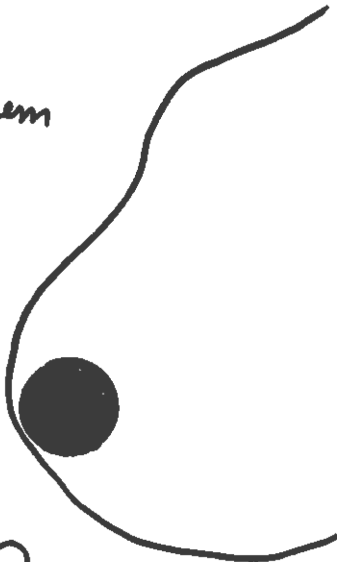
3°) HARNACK'S PRINCIPLE Suppose that each h_k is harmonic in Ω and that

$$h_1 \leq h_2 \leq h_3 \leq \cdots$$

Then $h = \lim_{k\to\infty} h_k$ is either harmonic or identically $+\infty$ in Ω.

4°) All derivatives of a harmonic function are harmonic. In $B(x,R)$ the solid Mean Value Property yields for u_{x_j}

$$\frac{\partial u}{\partial x_j} = \frac{1}{\mathrm{vol}(B_R)} \int_{B(x,R)} \frac{\partial u}{\partial x_j}\, dx$$



at the center x. By Gauss's theorem

$$\int_{B_R} \frac{\partial u}{\partial x_j}\, dx = \int_{\partial B_R} u \nu_j \, dS$$

Therefore

$$|\nabla u(x)| \le \frac{n \sup_{\Omega} |u|}{\mathrm{dist}(x, \partial\Omega)}, \qquad x \in \Omega$$

SUBHARMONIC FUNCTIONS

A function $u \in C^2(\Omega)$ is <u>subharmonic</u> if $\Delta u \ge 0$. It is essential to know that the pointwise maximum

$$u = u(x) = \max\{u_1(x), u_2(x)\}$$

of subharmonic functions is, itself, subharmonic. Thus one cannot, in general, require differentiability.

DEF.: The function $u : \Omega \longrightarrow [-\infty, \infty)$ is subharmonic in Ω, if

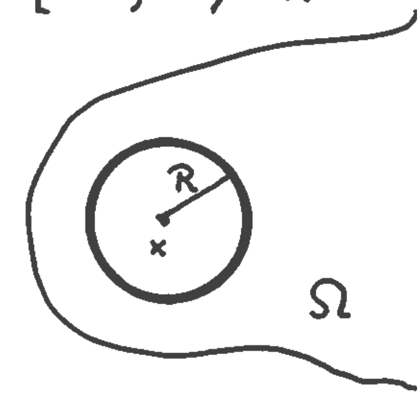
(i) $u \not\equiv -\infty$

(ii) u is upper semicontinuous

(iii) At each point $x \in \Omega$

$$u(x) \leq \frac{1}{\omega_n R^{n-1}} \oint_{|x-y|=R} u(y)\, dS(y)$$

whenever $0 < R < \operatorname{dist}(x, \partial\Omega)$.



submeanvalue property

<u>Ex.</u>: $\log|x|$ $(n=2)$ and $-|x|^{2-n}$ $(n \geq 3)$ are subharmonic in $\mathbb{R}^n$.

We shall consider only <u>continuous</u> subharmonic functions. This replaces (i) and (ii) above.

THEOREM A subharmonic function obeys the <u>Comparison</u> <u>Principle</u> in each subdomain $D \subset\subset \Omega$: if $h \in C(\overline{D})$ is harmonic in D and if $h|_{\partial D} \geq u|_{\partial D}$, then $h \geq u$ in D.

<u>Proof</u>: Also $u - h$ is subharmonic in D and by the Maximum Principle

$$\max_{\overline{D}} (u-h) = \max_{\partial D} (u-h) \leq 0$$

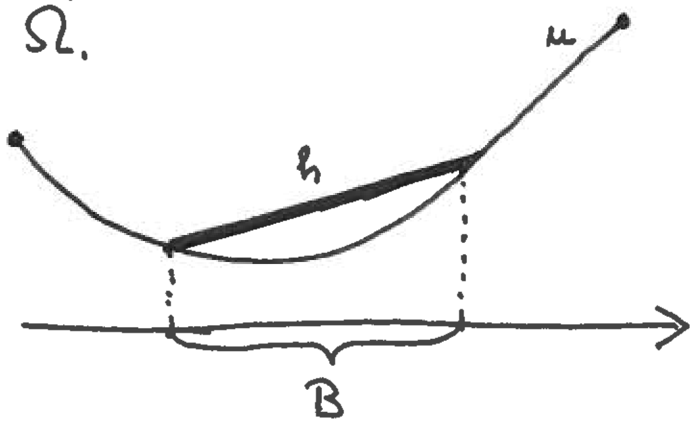
Hence $u - h \leq 0$ in D. $\square$

The POISSON MODIFICATION of a subharmonic function u is important for the Perron Method. Let $\overline{B}$ denote a ball in Ω. Define

$$U = \begin{cases} u & \text{in } \Omega \setminus B, \\ h = & \text{the Poisson integral of } u \\ & \text{over the sphere } \partial B \end{cases}$$

By the Comparison Principle $U \geq u$. Also U is subharmonic in Ω.

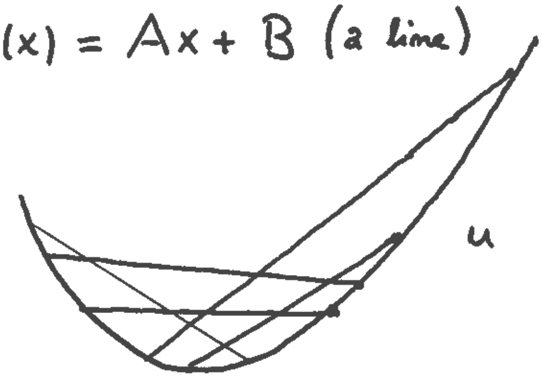
It clearly satisfies the Comparison Principle and that is enough.



We say that v is <u>superharmonic</u>, if $(-v)$ is subharmonic. "$\Delta v \leq 0$"; super mean value property.

The one dimensional case $n = 1$.

$$\begin{cases} \Delta u = 0 \text{ means } u'' = 0 \iff u(x) = Ax + B \text{ (a line)} \\ \text{subharmonic} = \text{convex} \\ \text{superharmonic} = \text{concave} \end{cases}$$



$u'' \geq 0$

PERRON'S METHOD FOR $\Delta H = 0$

The *lower class* $\mathcal{L}_f$ consists of those functions u that are subharmonic in Ω and satisfy

$$\lim_{x \to \xi} u(x) \leq f(\xi)$$

at each boundary point $\xi \in \partial\Omega$. (Also the function $u \equiv -\infty$ is member of the lower class, so that $\mathcal{L}_f$ is never empty.) The *upper class* $\mathcal{U}_f$ consists of those functions v that are superharmonic in Ω and satisfy

$$\lim_{x \to \xi} v(x) \geq f(\xi)$$

at each boundary point $\xi \in \partial\Omega$. (Also $\equiv +\infty$ is approved.). Define

$$\begin{cases} \underline{H}_f(x) = \sup\limits_{u \in \mathcal{L}_f} u(x), & \underline{\text{lower Perron solution}} \\ \overline{H}_f(x) = \inf\limits_{v \in \mathcal{U}_f} v(x). & \underline{\text{upper Perron solution}} \end{cases}$$

Then

$$-\infty \leq \underline{H}_f(x) \leq \overline{H}_f(x) \leq +\infty,$$

since $u \leq v$, if $u \in \mathcal{L}_f$ and $v \in \mathcal{U}_f$. Notice that, *if* there exists a function h that is harmonic in Ω and attains the right boundary values

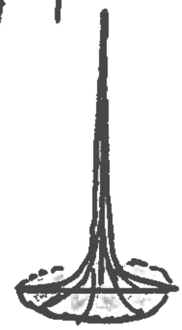
$$\lim_{x \to \xi} h(x) = f(\xi)$$

when $\xi \in \partial\Omega$, then

$$h(x) \underbrace{\le}_{h\in\mathcal{L}} \underline{H}_f(x) \le \overline{H}_f(x) \underbrace{\le}_{h\in\mathcal{U}} h(x)$$

so that $h = \underline{H}_f = \overline{H}_f$. In other words, the method "produces" the solution, if it exists.

EXAMPLE Let $\Omega = \{(x,y) \mid 0 < x^2+y^2 < 1\}$ be a punctured disk. The center $(0,0)$ is a boundary point. Define $f(x,y) = 0$ when $x^2+y^2=1$, and $f(0,0) = 1$. This is a continuous boundary function!



Now, for $\varepsilon > 0$,

$$0 \le \underline{H}_f(x,y) \le \overline{H}_f(x,y) \le \varepsilon \log\left(\frac{1}{x^2+y^2}\right)$$

when $(x,y) \in \Omega$, because $0 \in \mathcal{L}_f$ and the logarithm belongs to $\mathcal{U}_f$. As $\varepsilon \to 0$ we deduce that

$$\overline{H}_f = \underline{H}_f = 0.$$

The solution takes the wrong boundary value at the origin, which is an irregular boundary point. NOTE $\overline{H}_f$ itself does not belong to the upper class!

If $u_1, u_2, \dots, u_m \in \mathcal{L}_f$, then $\max\{u_1, u_2, \dots, u_m\}$ also belongs to $\mathcal{L}_f$. If $u \in \mathcal{L}_f$ so does its Poisson modification. Similar statements hold for the upper class.

Note that if f is bounded, so are the Perron solutions: $m \le f(\xi) \le M \Rightarrow m \le \underline{H}_f(x) \le \overline{H}_f(x) \le M$ (in this case $m \in \mathcal{L}_f$ and $M \in \mathcal{U}_f$). We can also assume that all functions u and v in the construction obey the same bounds (if $u \in \mathcal{L}_f$, $u \le M$ and $\max(u, m) \in \mathcal{L}_f$.)

THEOREM The function $\underline{H}_f$ is either harmonic in Ω, identically $-\infty$ in Ω, or identically $+\infty$ in Ω. A similar statement concerns $\overline{H}_f$.

<u>Proof</u> for $\underline{H}$ in the case $m \le f(\xi) \le M$. We may assume that any function in sight is bounded. Let $B_R = B(x_0, R)$ be an arbitrary ball in Ω, $\overline{B}_R \subset \Omega$. Let $x^1, x^2, x^3, \ldots$ be points in Ω. At the point x^j there is a sequence of functions u_k^j such that

$$\begin{cases} u_1^j \le u_2^j \le u_3^j \le \cdots, & \lim_{k\to\infty} u_k^j(x^j) = \underline{H}(x^j) \\ \text{and } u_k^j \in \mathcal{U}_f, & j = 1, 2, 3, 4, \ldots \end{cases}$$

Consider the sequence

$$\underset{w_1}{u_1^1}, \quad \underset{w_2}{\max\{u_2^1, u_2^2\}}, \quad \underset{w_3}{\max\{u_3^1, u_3^2, u_3^3\}}, \quad \underset{w_4}{\max\{u_4^1, u_4^2, u_4^3, u_4^4\}}$$

Then $w_1 \le w_2 \le w_3 \le \ldots$ and each $w_k \in \mathcal{U}_f$.

Moreover, $w_k(x^j) \longrightarrow \underline{H}(x^j)$ at each selected point x^j. Now, replace w_k by its Poisson modification

$$W_k = \begin{cases} w_k & \text{in } \Omega \setminus B(x_0, R) \\ h_k & \text{in } B(x_0,R) \quad = \text{ a harmonic function.} \end{cases}$$

Then $w_k \le W_k \le \underline{H}$ and $W_k \in \mathcal{L}_f$. By Harnack's theorem the increasing sequence $h_1, h_2, h_3, \dots$ converges to a function h that is harmonic in B_R. Thus

$$h \le \underline{H} \quad \text{in } B(x_0, R)$$

$$h(x^j) = \underline{H}(x^j) \quad \text{if } x^j \in B(x_0, R)$$

It is enough to prove that $h = \underline{H}$ in B_R. Notice that $\underline{H}$ is independent of how the points x^j are selected, but, in principle, the harmonic function h might depend on the points.

CLAIM $\underline{H}$ is continuous. – If $y_k \longrightarrow y$ we have to show that $\underline{H}(y_k) \longrightarrow H(y)$, assuming that the points y, y_k are in $B(x_0, R)$. Select $x^1 = y$, $x^2 = y_1$, $x^3 = y_2$ in the previous construction. Then

$$\underline{H}(y_k) = h(y_k) \longrightarrow h(y) = \underline{H}(y) \quad \text{as } k \longrightarrow \infty,$$

because h is continuous at the point y. This proves that $\underline{H}$ is continuous.

Now select the points x^j again so that, this time, $\{x^1, x^2, x^3, \dots\}$ is a dense subset of Ω. The points in Ω with rational coordinates will do fine. We arrive at the situation

$$h = \underline{H} \quad \text{in a } \underline{\text{dense}} \text{ subset !}$$

Since <u>both</u> functions are continuous, $h = \underline{H}$ in $B(x_0, R)$. In other words, $\underline{H}$ = a harmonic function in $B(x_0,R)$. The ball was arbitrary. This concludes the proof. ■

Finally, the question about the correct boundary values

$$\lim_{x \to \xi_0} \underline{H}_f(x) \overset{?}{=} f(\xi_0)$$

at a given boundary point ξ_0 can be reduced to the question about the existence of an auxiliary function, a so-called <u>barrier</u> function at ξ_0.

DEF Let $\xi_0 \in \partial\Omega$. The function $Q \in C(\overline{\Omega})$ is a barrier function at $\xi_0 \in \partial\Omega$, if

(i) Q is subharmonic in Ω

(ii) $Q(\xi) < 0$, when $\xi \in \partial\Omega$, $\underline{\xi \neq \xi_0}$.

(iii) $Q(\xi_0) = 0$.

By the Max. Prc. $Q < 0$ in Ω.

THEOREM $\lim_{x \to \xi_0} H_f(x) = f(\xi_0)$, if the barrier exists at ξ_0.

The Dead in the Classroom

by Steven Rushen
Penn State University
State College, Pennsylvania

The problem of when a person stops learning has received considerable attention. Many argue that people learn throughout their lives. Others assert that learning stops at an early age, and that any "learning" after that point is simply reapplying previous knowledge to fit a new situation. Many college professors believe that for most people learning stops sometime before a student's freshman year, giving further support to this second school of thought.

For my study I sided with the first school of thought. To an early morning freshman economics class of thirty live students, fifteen dead students were added and the effects were observed. After a full semester of careful study, the following observations were considered noteworthy. (See Table 1 for RIP[1] Coefficients.)

Attendance

On average, dead students are less likely to skip class than living students, especially on nice, warm days. Dead students had perfect attendance, were always in class early, and never left early (in fact they often stayed after and never complained when lectures ran long), unlike their living companions who had less than perfect attendance, were often tardy, and at times would leave early.

Behavior

On average, dead students were less disruptive than living students. Dead students are less likely to interrupt the instructor, be disrespectful, make noise, and ask irrelevant questions than their living counterparts.

Class Participation

There was no discernible difference between living and dead students' performances in class discussions, responses to questions from the instructor, or when called to the chalkboard to solve a problem.

Exam Performance

This seemed to be the weakest point of the dead students. On average their scores were 30 to 40 points below the class mean. The effect this had on the grade curve was substantial, as it pushed the grades of all of the living students up to a B+ or better.

Table 1.
Measures of "Relative Individual Participation"

Mean Student RIP Coefficients[1]

Category	Living	Dead
Attendance	0.56	1.00
Behavior	0.40	1.00
Participation	0.12	0.13
Exam Scores	0.45	0.09

Conclusion

It is the author's opinion that dead students definitely have a place in the classroom. Their perfect attendance and exemplary behavior clearly illustrate their desire to learn. In three of the areas described they were at least the equal of, if not superior to, their living peers. While their performance on exams was poorer than that of living students, this can not be taken as unwillingness to learn. The lower test scores could be due to low self-esteem, or to a misunderstanding, on the students' part, of general exam procedures. It is the author's opinion that in the near future "Outcome-Based Education" assessment may hold the key to overcoming this obstacle and give a better indication of the true learning ability of all students, vivacious or otherwise.

Note

1. RIP coefficients for Attendance and Exam scores are based on a straight percentage basis from performance in those respective categories. For Participation and Behavior this was based on both quantitative and qualitative measures of performance in these areas. Values of 1.00 equal a 100% or perfect performance, while 0.00 is 0%, or worst possible performance.