HEAT EQN.

$$\begin{cases} \boxed{u_t = k u_{xx}} & (-\infty < x < \infty,\ t > 0) \\ \lim_{t \to 0+} u(x,t) = f(x), & \text{initial temperature} \end{cases}$$

$$\hat{u}(\omega, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)\, e^{-i\omega x}\, dx$$

trans. with respect to x

$$\hat{u}(\omega, 0) = \hat{f}(\omega) \quad \text{OBS!}$$

$$\widehat{u_t(x,t)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial u(x,t)}{\partial t}\, e^{-i\omega x}\, dx$$

$$= \frac{\partial}{\partial t} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)\, e^{-i\omega x}\, dx \right\} = \frac{\partial}{\partial t} \hat{u}(\omega, t)$$

$$\widehat{u_{xx}(x,t)} = i^2 \omega^2 \hat{u}(\omega, t) \qquad (i^2 = -1)$$

The Eqn becomes

$$\boxed{\frac{\partial}{\partial t} \hat{u}(\omega, t) + k\omega^2 \hat{u}(\omega, t) = 0}$$

SOLUTIONS:

$$\hat{u}(\omega,t) = A(\omega)\, e^{-\omega^2 k t}.$$

$$\hat{f}(\omega) = \hat{u}(\omega,0) = A(\omega)\cdot 1$$

$$\hat{u}(\omega,t) = \hat{f}(\omega)\, e^{-\omega^2 k t}$$

$$= \frac{1}{\sqrt{2kt}}\, \hat{f}(\omega)\cdot \widehat{e^{-x^2/4kt}}$$

$$= \frac{1}{\sqrt{4k\pi t}}\, \widehat{f(x) * e^{-x^2/4kt}}$$

$$\boxed{u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} f(y)\, dy}$$

This is the only solution that remains bounded from below for all times t, for instance

$$u(x,t) \geq -273^\circ{,}15 .$$

There are "non-physical" solutions too.