

Ex. 3.3  $\Omega \subset \mathbb{R}^2$  a bounded domain and

$$\begin{cases} v_{xx} + v_{yy} = -2 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \quad v \in C^2(\Omega) \cap C^1(\bar{\Omega})$$

Claim:  $w \equiv |\nabla v|^2$  attains its maximum on the boundary.

Proof: 1)  $u = v + \frac{1}{2}(x^2 + y^2)$  is harmonic:  $\Delta u = 0$ .

2) The derivatives of harmonic functions are harmonic.

Thus  $u_x = v_x + x$ ,  $u_y = v_y + y$  are harmonic.

Then also  $v_x, v_y$  are harmonic functions ( $v_x = u_x - x$  = a difference of harmonic functions)

3)  $\Delta f = 0 \Rightarrow \Delta(f^2) = 2|\nabla f|^2 + 2f \Delta f = 2|\nabla f|^2 \geq 0$ . Thus  $\Delta(v_x^2) \geq 0$ ,  $\Delta(v_y^2) \geq 0$ .

Therefore

$$\Delta w = \Delta(v_x^2 + v_y^2) = \Delta(v_x^2) + \Delta(v_y^2) \geq 0 + 0 = 0$$

i.e.  $w = |\nabla v|^2$  is subharmonic.

4) The maximum principle holds for subharmonic functions. Hence

$$w = |\nabla v|^2$$

attains its maximum on the boundary  $\partial\Omega$ .

Remark  $v > 0$  in  $\Omega$ . Why?