NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for
TMA4305 Partial Differential Equations

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Examination time (from–to): 09:00–13:00

Permitted examination support material: One yellow A4-sized sheet of paper stamped by the Department of Mathematical Sciences. On this sheet the student may write whatever he wants. No other aids permitted.

Language: English

Number of pages: 2

Number pages enclosed: 0

Checked by:

Date Signature

Problem 1 Suppose that $u = u(x, y),\ v = v(x, y)$ satisfy the Cauchy-Riemann Equations

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

in the whole plane. Assuming that the second derivatives are continuous, calculate the Laplacian

$$\Delta(uv).$$

Which of the alternatives $\Delta(uv) > 0$, $\Delta(uv) = 0$, or $\Delta(uv) < 0$ holds ?

Problem 2 Consider the equation

$$u_t = u_{xx} + u_x^2 - 1, \qquad \text{in} \qquad Q_T = (-1, 1) \times (0, T).$$

Show that a solution with continuous second derivatives in $[-1, 1] \times [0, T]$ cannot attain its maximum anywhere else than on the *parabolic* boundary.

Problem 3 The energy integral for the Klein-Gordon Equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u + m^2 u \;=\; 0,$$

where m and c are constants and $u = u(x, y, z, t)$, is given by

$$E(t) = \frac{1}{2} \iiint_{\mathbb{R}^3} \left(u_t^2 + c^2 |\nabla u|^2 + m^2 u^2 \right) dx\, dy\, dz.$$

a) Suppose that $u \in C^2(\mathbb{R}^3 \times \mathbb{R})$ and that $u = 0$ when $x^2 + y^2 + z^2 \geq R^2$ (= some large number). Show that $E(t)$ is constant.

b) Show that the initial value problem

$$\begin{cases} u(x, y, x, 0) = f(x, y, z) \\ u_t(x, y, z, 0) = g(x, y, z) \end{cases}$$

for the Klein-Gordon Equation has at most one solution $u \in C^2(\mathbb{R}^3 \times [0, \infty))$ which is zero when $x^2 + y^2 + z^2 \geq R^2$.

Problem 4 Consider the weak solutions of the equation

$$2\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^4}{4}\right) = 0,$$

when $-\infty < x < \infty$ and $t > 0$.

a) Solve the initial value problem

$$u(x,0) = \begin{cases} 2 & \text{when} \quad x < 0 \\ 1 & \text{when} \quad x > 0. \end{cases}$$

Draw a picture of the shock curve and the characteristics.

b) Solve the initial value problem

$$u(x,0) = \begin{cases} 1 & \text{when} \quad x < 1 \\ 2 & \text{when} \quad x > 1. \end{cases}$$

Give an explicit formula for the rarefaction wave. Draw a picture in the xt-plane.

Problem 5 Let $v = v(x, y, x, t)$ denote the solution of the wave equation

$$v_{tt} = 9(v_{xx} + v_{yy} + v_{zz}) \qquad \text{in} \qquad \mathbb{R}^3 \times (0, \infty)$$

with inital values

$$v(x,y,z,0) = 0 \quad \text{and} \quad v_t(x,y,z,0) = \begin{cases} 1 - e^{-x^2-y^2} & \text{if} \quad z < 0 \\ 0 \quad \text{if} \quad z > 0. \end{cases} \quad .$$

When is $v(0,0,1,t) \neq 0$? Determine also

$$\lim_{t\to\infty} v(0,0,1,t).$$

Good luck!