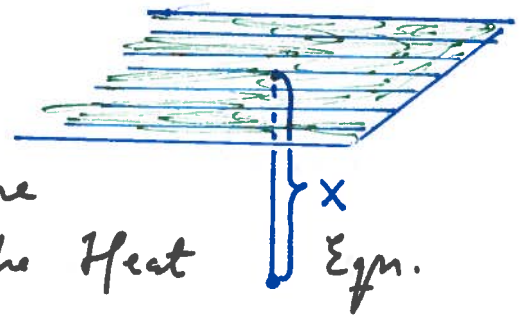


# THE PROBLEM OF THE EARTH'S TEMPERATURE

Treating the surface of the earth as a plane, we determine the temperature at depth  $x > 0$  from the Heat Eqn.



$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad (x > 0)$$

We have assumed that  $u(x, y, z, t) = u(x, t)$  so that  $\Delta u = u_{xx}$ . At the surface  $x = 0$  we have the boundary condition

$$u(0, t) = A \cos\left(\frac{2\pi t}{D}\right) + B \cos\left(\frac{2\pi t}{Y}\right) + C$$

due to daily and yearly variations; here  $Y \approx 365 D$ .

Let us first solve the equation with the surface temperature

$$u(0, t) = e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$$

Then the real part is the desired quantity.

Ansatz:  $u(x, t) = X(x)T(t)$

$$X T' = X'' T k$$

$$\frac{T'}{k T} = \frac{X''}{X} = \lambda \quad (\text{const. of separation})$$

We obtain

$$u(x, t) = c e^{k t \lambda} e^{\sqrt{\lambda} x}$$

calculate,  
please!

$$u(0, t) = c e^{k t \lambda} \stackrel{?}{=} e^{i \omega t}$$

Thus  $c = 1$ ,  $k \lambda = i \omega$ .

$$u(x, t) = \underline{e^{i \omega t} e^{\frac{\sqrt{i \omega}}{k} x}}$$

Now use

$$\sqrt{i} = \pm \frac{1+i}{\sqrt{2}}$$

It follows that (take  $\omega > 0$ )

$$u(x, t) = e^{i \omega t} e^{\pm (1+i) \frac{\sqrt{\omega}}{2k} x}$$

$$= e^{i(\omega t \pm \frac{\sqrt{\omega}}{2k} x)} e^{\pm \frac{\sqrt{\omega}}{2k} x}$$

The real part is

$$\cos\left(\omega t - \frac{\sqrt{\omega}}{2k} x\right) e^{-\frac{\sqrt{\omega}}{2k} x},$$

where we have discarded the + sign, since it leads to an unbounded temperature as  $x \rightarrow +\infty$ .

Superposition yields the final solution:

$$\begin{aligned}
 u(x,t) = & A e^{-\sqrt{\frac{\pi}{kD}} x} \cos\left(\frac{2\pi}{D} t - \sqrt{\frac{\pi}{kD}} x\right) \\
 & + B e^{-\sqrt{\frac{\pi}{kY}} x} \cos\left(\frac{2\pi}{Y} t - \sqrt{\frac{\pi}{kY}} x\right) \\
 & + C.
 \end{aligned}$$

↑  
 Phase lag  
 $\sqrt{\frac{\pi}{kY}} x$

The yearly term has the phase lag  
 $\sqrt{\frac{\pi}{kY}} x$

at depth  $x$ . At depth  $x$  coming from

$$\sqrt{\frac{\pi}{kY}} x = \pi, \quad x = \sqrt{\pi kY}$$

the temperature is totally out of phase from that on the surface:  $\cos\left(\frac{2\pi}{Y} t - \pi\right) = \text{MINUS} \cos\left(\frac{2\pi}{Y} t\right)$ .

Choosing a reasonable value of  $k$ , leads to a prediction that annual temperature changes will lag by six months at about 2-3 meters depth. — It is winter at a depth

of 2-3 meters when it is summer at the surface. (The amplitude is only a fraction of the surface amplitude.)