DIRICHLET'S PRINCIPLE

<u>The Dirichlet integral</u>

$$I(v) = \iiint_{\Omega} \underbrace{\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right]}_{|\nabla v|^2} dx\,dy\,dz$$

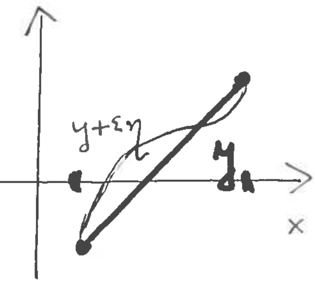
<u>The Laplace equation</u>

$$\boxed{\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{Euler-Lagrange Eqn.}}$$

<u>Dirichlet's Principle</u>: If the function u minimizes the Dirichlet integral among all functions v with the same boundary values, then

i.e. $I(u) \leq I(v)$, $v = u$ on $\partial\Omega$,

$$\Delta u = 0 \quad \text{in } \Omega.$$



<u>Proof</u> in the one-dimensional case

$\Omega = (a, b)$. Now

$$I(y) = \int_a^b y'(x)^2\,dx, \quad y \in C([a,b]) \cap C^1(a,b)$$

and $y(a) = A$, $y(b) = B$.

Let $\eta \in C([a,b]) \cap C^1(a,b)$, $\eta(a) = 0 = \eta(b)$.

If $y = y(x)$ is a minimizer, then

$$I(y) \leq I(y + \varepsilon\eta)$$

since $\underbrace{y(x) + \varepsilon\eta(x)}_{\text{COMPETING FUNCTION}} = y(x)$ when $x = a, b$.

Thus

$$\int_a^b y'(x)^2 dx \le \int_a^b y'^2(x)\,dx + 2\varepsilon\int_a^b y'(x)\eta'(x)\,dx + \varepsilon^2\int_a^b \eta'(x)^2 dx$$

PARABOLA

$$\equiv A + 2\varepsilon C + B\varepsilon^2 \quad (-\infty < \varepsilon < \infty)$$

Notice that the expression has its minimum at $\varepsilon = 0$ (regard ε as a variable) and therefore $C = 0$ or

$$\int_a^b y'(x)\eta'(x)\,dx = 0$$

LAPLACE'S
EQN IN 1 variable.

CLAIM: $y'(x) \equiv$ constant (Hence $y'' = 0$).

Let

$$\eta(x) = \int_a^x (y'(t) - C)\,dt, \quad C = \frac{1}{b-a}\int_a^b y'(t)\,dt$$

$\frac{y(b)-y(a)}{b-a}$

so that $\eta(a) = 0$, $\eta(b) = 0$. Now $\eta' = y'(x) - C$ and

$$\int_a^b \underbrace{(y'(x) - C)^2}_{\ge 0} dx = \int_a^b (y'(x) - C)\eta'(x)\,dx$$

$$= \int_a^b y'(x)\eta'(x)\,dx - C\int_a^b \eta'(x)\,dx = 0 - C\cdot 0 = 0.$$

But $(y'(x) - C)^2 \ge 0$. It follows that $(y'(x) - C)^2 \equiv 0$ and hence $y'(x) \equiv C$.

That a minimizer exists was first proved by D. Hilbert.

Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and that $\partial\Omega$ is regular enough (a ball, a cube, a Lipschitz domain will do) and that

$$f : \partial\Omega \longrightarrow \mathbb{R}$$

is continuous. Consider the problem of minimizing the Dirichlet integral $I(v)$ among all

$$v \in C(\overline{\Omega}) \cap C^2(\Omega), \quad v = f \text{ on } \partial\Omega.$$

Assume that $I(v) < \infty$ for at least one admissible v (necessary!!). Under these conditions there exists a minimizer u:

$$I(u) \leq I(v) \quad \text{for all admissible } v.$$

The minimizer is unique and $\Delta u = 0$.

The existence is difficult to prove. But the equation $\Delta u = 0$ comes easily. If u is a minimizer, consider the competing function

$$v = u(x) + \varepsilon\eta(x), \quad \eta \in C_0^2(\Omega)$$

Now

$$x = (x_1, x_2, \ldots, x_n)$$

$$\int_\Omega |\nabla u|^2 \leq \int_\Omega |\nabla u + \varepsilon \nabla\eta|^2$$

$$= \underbrace{\int_\Omega |\nabla u|^2 + 2\varepsilon \int_\Omega \nabla u \cdot \nabla\eta + \varepsilon^2 \int_\Omega |\nabla\eta|^2}_{\text{HAS A MINIMUM AT } \varepsilon = 0.}$$

$$\Longrightarrow \int_\Omega \nabla u \cdot \nabla\eta = 0$$

$$\Longrightarrow \int_\Omega \eta(\Delta u) = 0 \quad \text{for all } \eta \in C_0^2(\Omega)$$

Abbreviation: $\int_\Omega |\nabla u|^2 = \int\cdots\int_\Omega |\nabla u|^2 dx_1 dx_2 \cdots dx_n$.

$F = \nabla u$ later

LEMMA (Variational lemma) Let $F \in C(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a domain. If

$$(*) \qquad \int_\Omega F(x)\eta(x)\,dx = 0$$

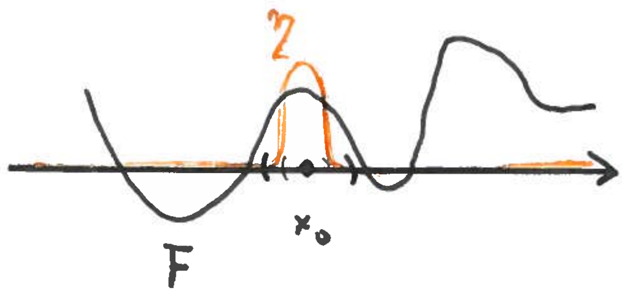
whenever $F \in C_0^\infty(\Omega)$, then $F(x) \equiv 0$ in Ω.

Proof: If there is a point $x_0 \in \Omega$ at which $F(x_0) > 0$, then, by continuity, $F(x) > 0$ in a (small) ball $B(x_0, \delta) \subset \Omega$. Choose η so that $\eta(x) > 0$, when $x \in B(x_0, \frac{\delta}{2})$, $\eta(x) = 0$, when $x \in \Omega \setminus B(x_0, \delta)$, and $\eta(x) \geq 0$ in Ω. Then

$$\int_\Omega F(x)\eta(x)\,dx = \int_{B(x_0,\delta)} F(x)\eta(x)\,dx \geq \int_{B(x_0,\frac{\delta}{2})} F(x)\eta(x)\,dx$$

$$> 0$$

This contradicts $(*)$. — The same goes for the case $F(x_0) < 0$. $\square$



Remark:

$$\eta(x) = \begin{cases} \dfrac{C}{\delta^n} e^{-\frac{\delta^2}{\delta^2 - |x-x_0|^2}}, & \text{when } |x-x_0| < \delta \\ 0, & \text{when } |x-x_0| \geq \delta. \end{cases}$$

Friedrichs's mollifyer