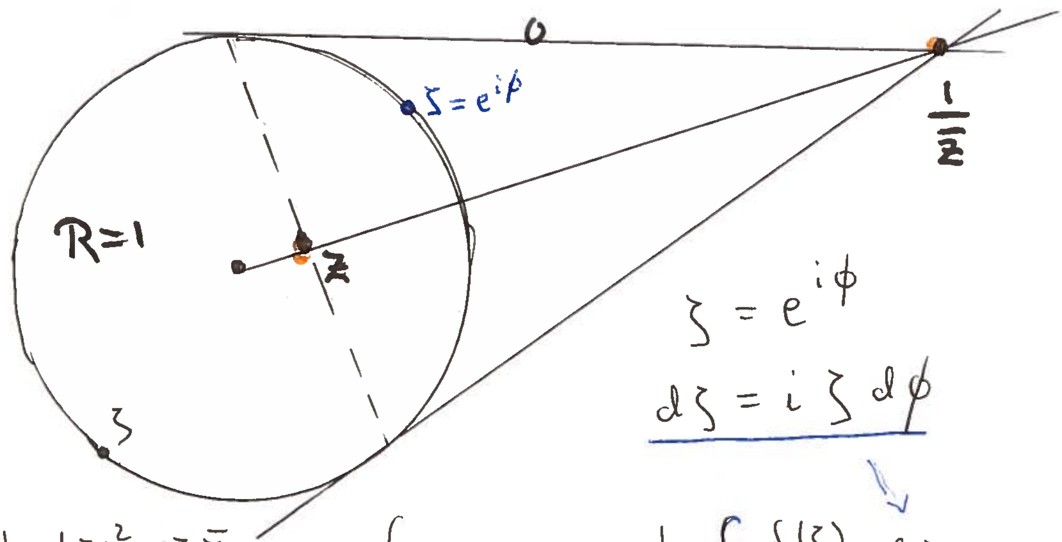
POISSON'S FORMULA FROM CAUCHY'S.

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2}\, u(R,\phi)\, d\phi$$

Boundary values on $|z|=R$.



$$\zeta = e^{i\phi}$$

$$d\zeta = i\,\zeta\, d\phi$$

$$f = \underline{u} + iv$$

$$1 = |\zeta|^2 = \zeta\bar{\zeta}$$

Cauchy

$$f(z) = \frac{1}{2\pi i}\oint_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \quad |z|<1$$

$$0 = \frac{1}{2\pi i}\oint_{|\zeta|=1} \frac{f(\zeta)}{\zeta - \frac{1}{\bar{z}}}\, d\zeta \qquad \left(\frac{1}{\bar{z}} \text{ is outside the contour}\right)$$

Subtract:

$$f(z) = \frac{1}{2\pi i}\oint_{|\zeta|=1} f(\zeta)\underbrace{\left[\frac{1}{\zeta - z} - \frac{1}{\zeta-\frac{1}{\bar{z}}}\right]}_{\frac{1}{\zeta - z} + \frac{\bar{\zeta}\bar{z}}{\bar{\zeta}-\bar{z}}} d\zeta$$

$$\frac{1}{\zeta-\frac{1}{\bar{z}}} = \frac{\bar{z}}{\zeta\bar{z}-1} = \frac{\bar{\zeta}\bar{z}}{1\cdot\bar{z}-\bar{\zeta}} \qquad (\zeta\bar{\zeta}=1)$$

$$= \frac{1}{2\pi \cancel{i}}\int_0^{2\pi} f(\zeta)\, \frac{\bar{\zeta}(1-|z|^2)}{|\zeta-z|^2}\, \cancel{i}\,\zeta\, d\phi \qquad (\zeta\bar{\zeta}=1)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|\zeta-z|^2}\, f(e^{i\phi})\, d\phi \qquad = u(z) + i\,v(z)$$

Take real parts and write $z = re^{i\theta}$

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{\underbrace{|e^{i\phi}-re^{i\theta}|^2}_{= 1-2r\cos(\theta-\phi)+r^2}}\, u(e^{i\phi})\, d\phi$$

This is Poisson's formula.