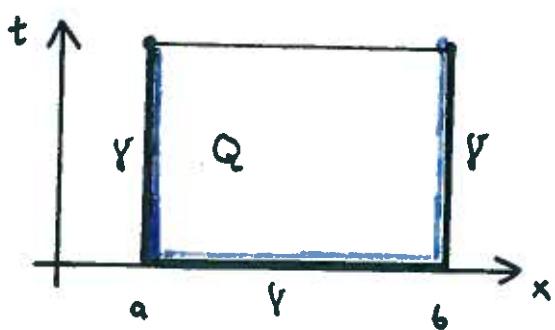


Maximum Principles for Parabolic Equations

If $u_{xx} - u_t > 0$ in a domain in the xt -plane, then u cannot have an interior maximum point. (Because $u_t = 0$ and $u_{xx} \leq 0$ at such a point. This contradicts the differential inequality.) Roughly speaking, this phenomenon is the cause of the parabolic maximum principles.

Let Q denote the open rectangle $a < x < b$ and $0 < t < T$. Its closure \bar{Q} is the closed set $a \leq x \leq b$, $0 \leq t \leq T$. The parabolic boundary Γ consists of three closed segments: $\{(x, 0) | a \leq x \leq b\}$, $\{(a, t) | 0 \leq t \leq T\}$, and $\{(b, t) | 0 \leq t \leq T\}$.



Suppose that

$$u \in C(\bar{Q}), u \in C^2(Q),$$

$$u_{xx} - u_t \geq 0 \text{ in } Q.$$

WEAK
MAX.
PRC.

If $u \leq 0$ on the parabolic boundary Γ , then $u \leq 0$ in \bar{Q} .

Proof: If this weak maximum principle is true for every smaller rectangle $[a, b] \times [0, T']$, $T' < T$, then the result follows for \bar{Q} by the continuity of u . Hence we may assume that

the derivatives u_{xx} and u_t exist also at the latest time $t = \bar{T}$, $a < x < b$. Define

$$v(x, t) = u(x, t) - \varepsilon t, \quad \varepsilon > 0$$

Then

$$v_{xx} - v_t = u_{xx} - u_t + \varepsilon \geq \varepsilon > 0$$

so that v cannot have an interior maximum point. Let $\max_{\bar{Q}} v = v(x_0, t_0)$. If $t_0 = \bar{T}$ and $a < x_0 < b$, then $v_{xx} \leq 0$ and $v_t \geq 0$ at this point. (Why?). But this is a contradiction to the inequality $v_{xx} - v_t \geq \varepsilon$. Hence $(x_0, t_0) \in \gamma$.

Thus $u(x_0, t_0) \leq 0$ by the original assumption. Now

$$\begin{aligned} u(x, t) &= v(x, t) + \varepsilon t \\ &\leq v(x_0, t_0) + \varepsilon t \\ &= u(x_0, t_0) + \varepsilon(t - t_0) \\ &\leq 0 + \varepsilon(t - t_0) \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $u(x, t) \leq 0$ Q.E.D.

Cor. (Uniqueness) Suppose that u_1 and u_2 belong to $C(\bar{Q}) \cap C^2(Q)$ and that ~~they~~ they are solutions to the heat equation $u_{xx} - u_t = 0$ in Q . If $u_2 \geq u_1$ on the parabolic boundary γ , then $u_2 \geq u_1$ in \bar{Q} .
If $u_2 = u_1$ on γ , then $u_2 = u_1$ in Q .

Proof: Consider the difference $u_2 - u_1$. \square

Remarks: 1) If $m \leq u(x,t) \leq M$, when $(x,t) \in \mathcal{V}$,

then $m \leq u(x,t) \leq M$ in \bar{Q} .

2) If $|u_2(x,t) - u_1(x,t)| \leq \varepsilon$, when $(x,t) \in \mathcal{V}$, then $|u_2(x,t) - u_1(x,t)| \leq \varepsilon$ in \bar{Q} .

This expresses stability with respect to boundary values on \mathcal{V} .

"subsolution"

"subcaloric"

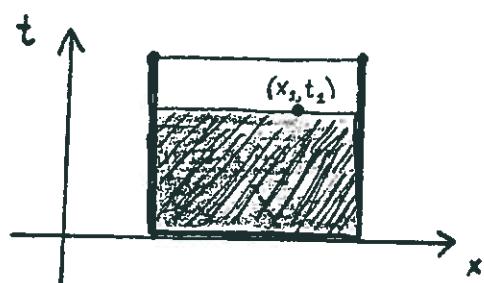
Theorem: Suppose that $u_{xx} - u_t \geq 0$ in Q and that $u \in C(\bar{Q}) \cap C^2(Q)$. Let $M = \max_{\bar{Q}} u$.

If $M = u(x_1, t_1)$ at some interior point (x_1, t_1)

$a < x_1 < b$ and $0 < t_1 < T$ or at some point

$a < x_1 < b$ and $t_1 = T$, then $u(x, t) = M$, when

$a \leq x \leq b$ and $0 \leq t \leq t_1$. (In other words, u attains its maximum at all "earlier points".)



If $u(x_1, t_1) = \max u$,
then $u \equiv M$ in the shaded
rectangle.

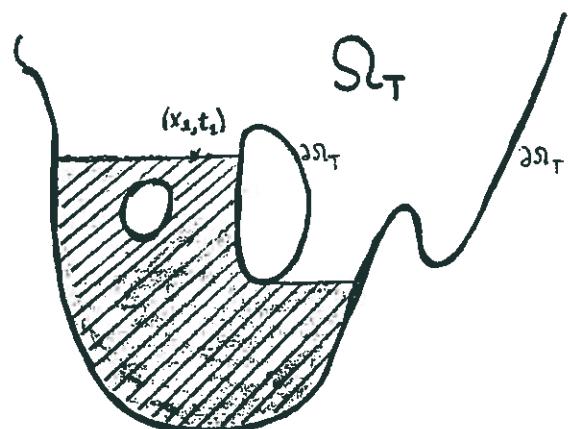
Proof: See Murray Protter &
Hans Weinberger, Maximum
Principles in Differential Eqs,
1967, 1984.

For more general domains
in the (x,t) -plane the
corresponding picture looks as
follows:

Principle



\downarrow $(N \cup T)$



This domain has 3 boundary components

A variant of the proof of the parabolic MAXIMUM PRINCIPLE

THEOREM Let $u \in C^2(\Omega \times (0, T))$, $u \in C(\bar{\Omega} \times [0, T])$. If $\Delta u \geq u_t$ in $\Omega \times (0, T)$, then the maximum of u is attained on the parabolic boundary, provided that Ω is bounded.

"SUBSOLUTION"

Proof: Let $\varepsilon > 0$ and consider

$$v = u - \frac{\varepsilon}{T-t}.$$

$$\Delta v = \Delta u, \quad v_t = u_t - \frac{\varepsilon}{(T-t)^2}, \quad v(x, T) = -\infty$$

$$\Delta v \geq v_t + \frac{\varepsilon}{(T-t)^2} \geq v_t + \frac{\varepsilon}{T^2} > v_t$$

*)

Therefore v cannot attain an interior maximum and the points when $t = T$ are out of the question. Thus $v(x, t) \leq \max_T v$, where T is the parabolic boundary. Thus

$$u(x, t) - \frac{\varepsilon}{T-t} \leq v(x, t) \leq \max_T v \leq \max_T u$$

$$u(x, t) \leq \max_T u + \frac{\varepsilon}{T-t}$$



Now, let $\varepsilon \rightarrow 0+$ and conclude that $u(x, t) \leq \max_{\Gamma} u$.

*) At an interior maximum point $\Delta v \leq 0$ and $v_t = 0$, by Calculus.