TMA4295 Statistical inference

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Lecture 2 in week 42: Interval and set estimation

Contents

1 Interval/set estimation

Definition 1.1 (Interval estimation) An interval estimator is an interval-valued estimator.

Example 1.1 Bayes credibility interval

An interval with posterior probability $\geq 1 - \alpha$ (95%)

$P(\tau(\theta) \in \hat{\tau}(x) \,|\, X = x) \geq 1 - \alpha$

$\hat{\tau} = [\hat{\tau}_1, \hat{\tau}_2]$ where $\hat{\tau}_1$ and $\hat{\tau}_2$ are real valued statistics.

Example 1.2 Confidence interval with level $1 - \alpha$

$P(\tau \in \hat{\tau}(X)) \geq 1 - \alpha$,where $P(\tau \in \hat{\tau}(X))$ is the coverage probability (parameter)

Example 1.3 Prediction interval with level $1 - \alpha$

$P(T \in \hat{\tau}(X)) \geq 1 - \alpha$, where T is a random point and $\hat{\tau}(X)$ is a random interval.

(T,X) has known distribution when θ is known.

What is a best/optimal intervall?

Loss function: $l = |\hat{\tau}| \; + \lambda(\tau \notin \hat{\tau})$ where $\lambda > 0$ (Lagrange multip.)

Clases of set estimators:

- Unbiased: Coverage probability of true-value is always larger then coverage probability of false value.
- Equivariance: $\hat{\tau}(gx) = g\hat{\tau}(x)$
  $[g\hat{\tau} = \{gt \,|\, t \in \hat{\tau}\}]$
  $\hat{\tau}(x + a) = a + \hat{\tau}(x)$
  Testing: $l = \tau(1 - t) + \lambda(1 - \tau)t$ is similar to above loss

$\rho = E[l(\hat{\tau}(X), \tau)] \leq \rho' \Leftrightarrow \hat{\tau}$ is better than $\hat{\tau}'$

Theorem 1.1 (Kolmogorov-Robbins Theorem) $(a, w) \mapsto (a \in A(w))$ is measurable.
E($\mu(A)$) = Expected size of random set A
= $\int$ P(a $\in A)\mu$(da) , where $P(a \in A)$ is the hitting probability

Proof:

R.H = $\int[\int$(a)$\in A(w))P(dw)]\mu(du)$

= $\int[\int$(a)$\in A(w))\mu(da)]P(dw)$

= $\int\mu(A, w)P(dw)$

= E $\mu(A)$ $\square$

$\rho = E\mu(\hat{\tau}(X)) + \lambda P(\tau \notin \hat{\tau}(X))$, where $P(\tau \notin \hat{\tau}(X)) = \alpha$

= $\int P(t \in \hat{\tau}(X)\mu(dt)) + \lambda\alpha$ Assume that $\mu\{\tau\} = 0$

= $\int_{t \neq \tau} P(t \in \hat{\tau}(X))\mu(dt) + \lambda\alpha$ gives:

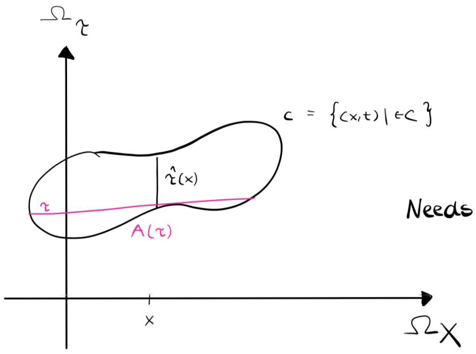
Theorem 1.2 (Ghosh-Pratt Theorem) A uniformly most powerfull set of a given level $1 - \alpha$ is optimal.

(The opposite is not true!)

Claim : There exists confidence sets in the class of level $1 - \alpha$ with smallest expected size without being UMP.

UMP: uniformly most powerfull.

Inversion of tests:



$\hat{\tau}(x) = \{\tau \,|(x, \tau) \in C\}$ = Confidence set

$A(\tau) = \{x \,|(x, \tau) \in C\}$ = Acceptance region

> **Theorem 1.3 (Test Inversion)** $\hat{\tau}(X)$ is $1-\alpha$ confidence set if and only if $A(\tau_0)^c$ is a α level test of $H_0 : \tau = \hat{\tau}$

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