Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4295 Statistical inference

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Lecture 1 in week 42: 'Hypothesis Testing'

Contents

1 From last time 2

2 How to define inverse of CDF if CDF is not monotonical 3

3 Reject an unlikely hypothesis! 4

4 Fisher solution 5

5 Solution using exponential families 5

Definitions 8

Theorems 8

Examples 8

Index 9

1 From last time

Firstly we recall from last time about hypothesis testing:

Definition 1.1 (Test) A test is a $\{0, 1\}$ valued statistic.

Definition 1.2 (Hypothesis) A hypothesis is a $\{0, 1\}$ parameter.

Loss function for tests:

$$l = \tau(1 - t) + (1 - \tau)tw$$

Where l is loss, τ is hypothesis, t is test, w is some (typical large) weight. Power of test T:

$$\beta = E(T)$$

This is parameter: $\beta = \beta(\theta)$. $\beta \leq \alpha$ on H_0 (which defines α)

β is power function. All information about test is in β. If we want to calculate the risk:

$$\rho = E(l(\tau, T)) = \tau(1 - \beta) + (1 - \tau)\beta w$$

then the problem of minimizing the risk comes down to maximize β.

Neymann-Pearson test - randomized test for two models. Gives Karlin-Rubin thm. Sufficiency and MLR is key. MLR implies monotonic power function which means that:

$$T_0 \prec T_1$$

, for $\theta_1 \leq \theta_0$

Definition 1.3 (stochastic order) A random variable is stochastic larger (or smaller) than other random variable if and only if it's cdf (or cmf) is larger or equal than other. In other words:

$$X_1 \prec X_2 \Leftrightarrow F_1 \geq F_2$$

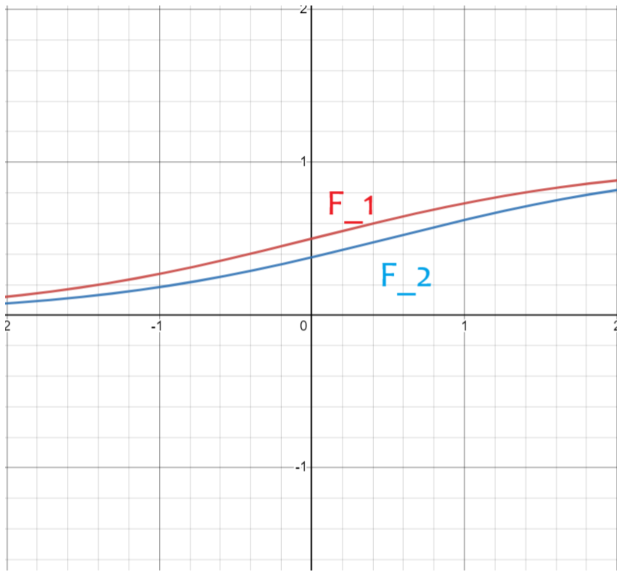

Figure 1: Graph for Definition 1.3

Theorem 1.1 (stochastic order) $X_1 \prec X_2 \Leftrightarrow \exists \widetilde{x}_i \sim X_i \sim x_i, \widetilde{X_1} \leq \widetilde{X_2}$

Proof. (how we should construct $\widetilde{x}_2$)

$\widetilde{x}_i = F_i^{-1}(U) \sim x_i$, where $U \sim U(0,1)$. This implies that $\widetilde{x}_1 \leq \widetilde{x}_2$ because of CDF.

Reminder that: $X \leq Y$ *means* $X(\omega) \leq Y(\omega) \forall \omega$ □

2 How to define inverse of CDF if CDF is not monotonical

$$F_-^{-1}(u) = inf\{x|u \leq F(x)\}$$

Note: we could also write above $F^{-1}(\omega) = min(x|u \leq F(x))$

$$F_+^{-1}(u) = sup\{x|F(x) \leq u\}$$

$$F^{-1}(u) = pF_-^{-1}(u) + qF_+^{-1}(u)$$
$$X \sim F$$

Where $p + q = 1$. In any case $X \sim F^{-1}(U), U \sim U(0,1)$.

Theorem 2.1 (consequence of theorem 1.1) $F(X) \succ U(0,1)$ when F is the CDF of X.

Proof.

$$F(F^{-1}(U)) \geq U$$

$$F(F^{-1}(u)) \geq u$$

□

3 Reject an unlikely hypothesis!

Let

$$t = (\lambda(x) \leq \lambda_\alpha)$$

Where t is the test, x is data, $\lambda(\cdot)$ is likelihood of hypothesis, $\lambda(x) = \frac{\hat{L_0}}{\hat{L}}$, $\hat{L_0}$ is $\sup\{L\}$ given H_0 and $\hat{L}$ is $\sup\{L\}$.

p value corresponding to this is by definition:

Definition 3.1 (p-value of a likelihood test)

$$p(x) = \sup_{H_0} P(\lambda(X) \leq \lambda(x))$$

Theorem 3.1 (this p is a p-value statistic.)

Proof. T 2.1 □

4 Fisher solution

Lets consider a conditional test given by conditioning on a statistic which is sufficient given H_0.

Fisher:

$$S = X_1 + X_2 \overset{H_0}{\sim} B(n_1 + n_2, p_i)$$

Where $X_i \sim B(n_i, p_i)$.

$X_1|S$ has a known distribution. The question is: what is reasonable rejection boundary? One answer to this question is to reject if X_1 is large.

$$p(x) = p(s, x_1) = P(X_1 \geq x_1|S = s)$$
$$P(p(X) \leq \alpha|S = s) \leq \alpha$$
$$E(P(p(X) \leq \alpha|S = s) \leq \alpha) \leq \alpha$$

So, p is a p-value statistic.

5 Solution using exponential families

Let

$$f(x) = h(x)e^{\theta t - \gamma}$$

Lets consider H_0 such that:

$$H_0 : \theta_1 = \theta_1^*$$
$$H_1 : \theta_1 \neq \theta_1^*$$

Where θ_1^* is some fixed value. θ_2 is nuisance here.

$$\theta = (\theta_1, \theta_2)$$

Solution:

Condition on $S = T_2$, then $X|S$ is known under H_0. Conditional p-value for this problem:

$$p(x) = p(t_1, t_2), P(p(T_1, t_2) \leq \alpha | t_2) \leq \alpha$$
$$\beta \leq \beta_0$$
$$P(p(x) \leq \alpha) = E(P(p(T_1, T_2) \leq \alpha | T_2))$$

The role of condition here is that point H_0 is controlled conditionally.

Example 5.1 (Exponential distribution) Let $x_1, x_2, \ldots, x_n \sim Exp(\beta)$. The likelihood then is:

$$L = \prod_{i=1}^{n} \frac{1}{\beta} e^{-\frac{x_i}{\beta}} = \beta^{-n} e^{-n\frac{\overline{x}}{\beta}}$$

By sufficiency principle all inference can be made based on $\overline{x}$, so:

$$\overline{x} = \beta \overline{u}, u_i \sim Exp(1)$$

.

The above is data generating equation. The $\overline{u}$ here is $\Gamma(n, 1/n)$.

Let's calculate likelihood ratio. Let:

$$H_0 : \beta \leq \beta_0$$

$$H_1 : \beta > \beta_0$$

then $\lambda = \frac{L_0}{\hat{L}} = \ldots$, which is possible to compute, but we can use Karlin-Rubin which is simpler because we have 1 parameter.

We consider $\frac{L_1}{L_0}$ which is increasing in $\overline{x}$ when $\beta_1 > \beta_0$

$$Exp(\beta_1) > Exp(\beta_2)$$

if $\beta_2 \geq \beta_1$

$e^{-n\overline{x}(\frac{1}{\beta_1} - \frac{1}{\beta_0})}$ is increasing as a function of $\overline{x}$.

Optimal test:

$$t = (\overline{x} > x_\alpha)$$

The next question is: how to get x_α?

$$\alpha = P(\overline{X} > x_\alpha)$$

Where x_α is critical value for $\Gamma(n, \frac{\beta_0}{n})$. If we find x_α, then we have optimal test.

Observation: We can use tables for χ^2!

Best solution: $p = P_0(\overline{X} > \overline{x}) = \{\text{using } \chi^2\} = \dots$

Example 5.2 (Two sided case) Let:

$$H_0 : \beta = \beta_0$$

$$H_1 : \beta \neq \beta_0$$

Then we don't have Karlin-Rubin.

Solution: We have to calculate $\lambda(x)$, and then p.

Definitions

1 **Test** 2
2 **Hypothesis** 2
3 **stochastic order** 2
4 **p-value of a likelihood test** 4

Theorems

1 **stochastic order** 3
2 **consequence of theorem 1.1** 4
3 **this** p **is a p-value statistic.** 4

Examples

1 Exponential distribution 6
2 Two sided case 7

Index

Hypothesis, **2**

p-value of a likelihood test, **4**

stochastic order, **2**

Test, **2**