Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4295 Statistical inference

Lecturer Fall 2023: *Gunnar Taraldsen*

Scribe: *Cassandra Jan*

Lecture 1 in week 41: 'Hypothesis Testing'

Contents

1 Basic concepts

Definition 1.1 (Hypothesis) A hypothesis is an event in the model space.

A hypothesis is an indicator of the model, so it is a parameter.

Definition 1.2 (Test) A test is a statistic with Bernoulli distribution.

Definition 1.3 (Power of a test) The power of a test T is defined as :

$$\beta = \mathbb{E}[T]$$

In the definition, β is a function of θ. It represents the probability of rejection of H_0. It is a parameter.

Note : τ is a 0-1 value parameter and T is a 0-1 valued estimator.

$\tau(\theta) = (\theta \in H1)$ where $\Omega_\theta = H_0 \cup H_1$, with $H_1 = H_0^c$ and $H_0 =$ Null hypothesis.

So for example if we have $H_0 : \theta = \theta_0$ and $H_1 : \theta \neq \theta_0$,
Then $H_0 = \{\theta | \theta = \theta_0\} = \{\theta_0\}$.

Likewise in the case of a normal distribution if we have: $H_0 : \mu = \mu_0$ and $H_1 : \mu \neq \mu_0$,
Then $H_0 = \{(\mu, \sigma) | \mu = \mu_0\}$.

Definition 1.4 (Level of a test) T has level α if $\beta \leq \alpha$ for $\theta \in H_0$.

Saying that T is an α-size test means that:

$$\sup_{\theta \in H_0} \beta(\theta) = \alpha$$

Therefore, we compare tests by comparing power function.
Risk (which represent the expected loss) is given uniquely by power function.
You know the distribution of T if you know the power function.

2 Work with hypothesis testing

Definition 2.1 (Likelihood of a hypothesis) The likelihood of H_0 is :

$$\lambda(x) = \frac{sup_{H_0} L}{sup L}$$

where L is the likelihood and x is the data.

We often have : $\sup_{H_0} L = L(\hat{\theta}_0)$ and sup L = L($\hat{\theta}$) where L($\hat{\theta}_0$) represent the maximum likelihood function given that we have H_0.

By definition, if λ is small we reject H_0.

Definition 2.2 (Likelihood of a test) It is defined by :

$$t = (\lambda \leq \lambda_0)$$

where λ is a statistic.

The size of the test is

$$\alpha = \sup_{\theta \in H_0} \ \beta(\theta) = \sup P(\lambda(X) \leq \lambda_\alpha)$$

VERY IMPORTANT NOTION : We reject an unlikely hypothesis

Definition 2.3 (Randomized test) A randomized test T is a statistic with

$$0 \leq T \leq 1.$$

Theorem 2.1 (Neymann-Pearson) The theorem is illustrated by the following figure.

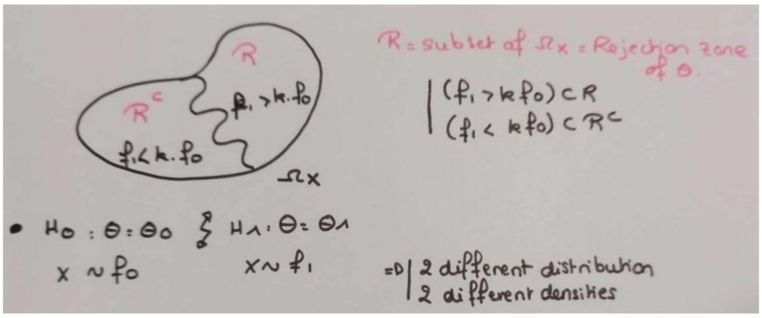

Figure 1: Neymann-Pearson optimal test R

Here k gives the level and defines R. The test R is optimal in the class of randomized test with level $\alpha = \beta_0 = E_0(R)$.

Note : We have $\beta_1 \geq \widetilde{\beta_1}$ (for all $\widetilde{R}$ test with $E_0\widetilde{R} \leq \alpha$) and $\beta_0 \leq \beta_1$ is a consequence.

Proof : (R-$\widetilde{R}$)$(f_1 - k.f_0) \geq 0$.

(Remark : if you integrate it, you still get something positive).

And then $\beta_1 \geq \widetilde{\beta_1} + k.(\beta_0 - \widetilde{\beta_0}) \geq \widetilde{\beta_1}$

3 Different possible tests

(i) Likelihood test (often better than (ii))

(ii) "Bayes" Rejected if the posterior probability of H_0 is small.

(iii) H_0 : $\tau = \tau_0$ (here : τ is a function of θ) ; H$_1$:$\tau \neq \tau_0$ We reject if T is far from τ_0 with T the estimator of τ. (It will depend on standard uncertainty).
This test is intuitively what we would do.

Remark : "Optimal" means "uniformely larger powered in class of tests with level α".

Definitions

Theorems

Examples

Index