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TMA4295 Statistical inference

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**Lecture 1 in week 40:
'Standard Uncertainty'**

Contents

1	Fundamental Concepts	2
2	Standard Uncertainty	2
3	Lower bound of the standard deviation	3
4	Fisher Information Metric	4
	Definitions	5
	Theorems	5
	Examples	5
	Index	6

1 Fundamental Concepts

Definition 1.1 (Statistic) A statistic is a function of data.

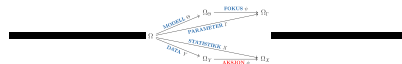
Definition 1.2 (Parameter) A parameter is a function of model.

2 Standard Uncertainty

Definition 2.1 (Standard Uncertainty) Standard uncertainty u is an estimate of the standard deviation of the estimator.

One should always state the Standard Uncertainty!

Example 2.1 (Uncertainty of the length of the NTNU pendulum) The length of the NTNU pendulum is $25.26(1)m$, where $25.26m$ is the estimate of the length, and (1) is the standard uncertainty. In this context, $25.26(1)m = 25.26m \pm 0.01m$.



Example 2.2 (Estimators of the length of the NTNU pendulum) The length of a pendulum is given by,

$$\lambda = \left(\frac{\tau}{2\pi}\right)^2 g, \tag{1}$$

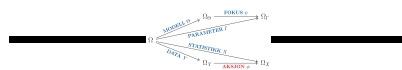
where τ is the period and g is the earths gravitational pull. Two possible estimates of this parameter are;

$$\hat{\lambda} = \left(\frac{\bar{t}}{2\pi}\right)^2 g, \tag{2}$$

$$\lambda^* = \bar{x}, \text{ where } x_i = \left(\frac{t_i}{2\pi}\right)^2 g, \tag{3}$$

These two estimators are two different instruments used for estimating the length. So the question is, which one do we pick?

To decide this, we need a specific statistical model. One possibility could be that $t_1, \dots, t_n \sim \mathcal{N}(\mu, \sigma^2)$. Then the best estimator would be the one that is minimally sufficient. In this possibility, the common minimally sufficient statistics are \bar{t} and s . Since $\hat{\lambda}$ is the only estimator that depends on \bar{t} , it is the best one of the two. λ^* does not, but maybe this estimator could give a smaller bias or variance than $\hat{\lambda}$. The Rao-Blackwellization, $\lambda^{**} = E[\lambda^*(T)|\bar{t}, s(t)]$, of λ^* could give us a third estimator, and this estimator would be better than λ^* .



How do we find the optimal estimator of λ ? The distribution of λ is found through,

$$\bar{t} = \tau + \sigma \bar{z}, \text{ where } z_i \sim \mathcal{N}(0, 1), \tag{4}$$

$$\bar{t}^2 = \tau^2 + 2\sigma\tau\bar{z} + \sigma^2\bar{z}^2, \tag{5}$$

$$E[\bar{T}^2] = \tau^2 + \sigma^2 E[\bar{z}^2], \tag{6}$$

$$E[\bar{T}^2 - ks^2] = \tau^2. \tag{7}$$

This gives UMVU (uniformly minimum-variance unbiased) and UMRU (uniformly minimum-risk unbiased) estimators of λ :

$$\check{\lambda} = \frac{g}{(2\pi)^2} (\bar{t}^2 - ks^2). \tag{8}$$

3 Lower bound of the standard deviation

The Cramer-Rao inequality gives us,

$$Var[T] \geq (\tau')^2 \iota^{-1}. \text{ where } \iota = Var[S] = Var[\partial_{\theta} \ln(f(X))]. \tag{9}$$

So $\sqrt{Var[T]} \geq \sqrt{\tau'^2 \iota^{-1}}$ gives the lower bound of the standard deviation of T , and T is unbiased. This is easier to calculate than the standard deviation. One idea could be

to use the Cramer-Rao inequality to report the standard uncertainty, but this would be optimistic. The ideal case would be that $\tau(\theta) = a + b\theta$, then $\tau(\hat{\theta}) = a + b\hat{\theta}$. Then,

$$\text{Var}[\tau(\hat{\theta}(X))] = b^2 \text{Var}[\hat{\theta}(X)]. \quad (10)$$

Here, $\tau(\hat{\theta}(X))$ is only unbiased if $\hat{\theta} \sim \mathcal{N}(0, \sigma)$, and then $\text{Var}[\hat{\theta}(X)] = \iota^{-1}$

4 Fisher Information Metric

Let $X \sim f$, $R(X) = \{x_1, \dots, x_m\}$, X is a simple random point. $f(x) = P(X = x)$, and f is unknown, $\sqrt{f} \in \mathfrak{R}^m$, $|\sqrt{f}| = 1$.

Assume P_X is known when time t is known. When $m = 3$ we have the unit ball. Distance is given by speed and time, $d(t_1, t_2) = \int_{t_1}^{t_2} v dt$. Then,

$$\begin{aligned} v^2 &= \sum_{i=1}^m (\partial_t \sqrt{f(x_i)})^2, \text{ where } \partial_t \sqrt{f} = \frac{1}{2} \partial_t \ln(f) \cdot f^{\frac{1}{2}} \\ &= \sum_{i=1}^m \frac{1}{4} (\partial_t \ln(f))^2 f \\ &= \frac{1}{4} E[S^2], \text{ where } S \text{ is the score function.} \end{aligned}$$

$$v = \frac{1}{2} \sqrt{\iota}, \text{ where } \iota = E[S^2] = \text{Var}[S].$$

So, $d(t_1, t_2) = \frac{1}{2} \int_{t_1}^{t_2} \sqrt{\iota} dt$. In conclusion, the Fisher Information gives both a method for calculating Standard Uncertainty and also a metric on the model space.

Definitions

1	Statistic	2
2	Parameter	2
3	Standard Uncertainty	2

Theorems

Examples

1	Uncertainty of the length of the NTNU pendulum	2
2	Estimators of the length of the NTNU pendulum	2

Index

Parameter, [2](#)

Standard Uncertainty, [2](#)

Statistic, [2](#)