Norwegian University of Science and Technology
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TMA4295 Statistical inference

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Lecture 2 in week 36: 'Symmetry, group actions and group family'

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1 Symmetry: Group actions

The topic of this lecture is symmetry. For this principle, one needs the concept of groups from mathematics. This and the equivariant parameters in statistics are defined and explained in more detail below.

> **Definition 1.1 (Equivariance of a statistic)**
> A statistic t is equivariant if $t(gx) = gt(x)$, where x is the data and g is the group action.

In this definition, the g from the left is an element of the group and gx is a one-to-one mapping of the data. Further, the g of the right-hand side of the equation is a function of the action on the range of the statistic.

A comment on the equation: The group action is linked to the action.

> **Definition 1.2 (Equivariance of a parameter)**
> A parameter τ is equivariant if $\tau(g\theta) = g\tau(\theta)$, where θ is the model and g is the group action.

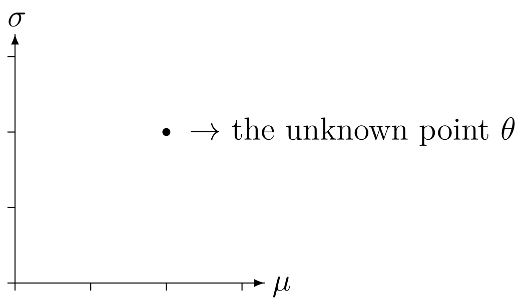
Now follows an example of these definitions, which deals with location-scale groups.

Example 1.1 (Location-scale Group) Let the data be $x_i = \mu + \sigma z_i$, $i = 1, \ldots, n$ where z has a known distribution. Further, z is a random variable and the components need not be independent. Assume $z \in \mathbb{R}^n = \Omega_x$. The parameter $\theta = (\mu, \sigma) \in \Omega_\theta = H$, with Ω_θ as the model space. Further, $\Omega_\theta = G$ is actually a group.

A little reminder: $z_i = Z_i(\omega)$ with $\omega \in \Omega$.

Question about the example (Possible exam question): Is $\{\omega\}$ an event?
Answer: No, it need not to be an event. (Remember: Ω is never specified.)

The following is a small sketch of the current example:



2 Group operations

This chapter first deals with a definition of the location-scale group and then with group operations.

Definition 2.1 (Location-scale group)
A location-scale group is given by $g(x) = a + bx$ with $g : \mathbb{R} \to \mathbb{R}$, where $a \in \mathbb{R}$ the location parameter and $b > 0$ as the scale parameter. Further, $(a,\ b) \in H$.

Example 2.1 (Inverse function of the location-scale group) In this example, the calculation of the inverse function is briefly presented using the example of the location-scale group.

$$
\begin{aligned}
&a + bx = y \\
\Leftrightarrow\ & x = \frac{y - a}{b} = \frac{1}{b}y - \frac{a}{b}
\end{aligned}
$$

Then $(a,\ b)^{-1} = (-\frac{a}{b},\ b^{-1})$.

Theorem 2.1 (Inverse of a product)
An inverse function from a composition of functions $g = g_1 \circ g_2$ is given by $g^{-1} = g_2^{-1} \circ g_1^{-1}$.

Theorem 2.2 (Cayley's Theorem: All groups are transformation groups)
All groups may be represented as a group of one-to-one transformations with group operation given by composition of functions.

Axiomes:

(i) $g^{-1}g = gg^{-1} = e$,

(ii) $eg = ge$,

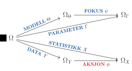
(iii) $(ab)c = a(bc)$.

Example 2.2 (Group operation - multiplication) This is an example of a group operation.

$$\begin{aligned} g_1 \circ g_2(x) &= g_1(a_2 + b_2 x) \\ &= a_1 + b_1(a_2 + b_2 x) \\ &= a_1 + b_1 a_2 + b_1 b_2 x \end{aligned}$$

Here $a_1 + b_1 a_2$ is the first component and $b_1 b_2$ the second component. Further applies $(a_1,\ b_1) \in H$ and $(a_2,\ b_2) \in H$, with $a_1,\ a_2 \in \mathbb{R}$ and $b_1,\ b_2 > 0$.

The realisation follows $(a_1,\ b_1) \circ (a_2,\ b_2) = (a_1 + b_1 a_2,\ b_1 b_2)$. It is also worth noting that $e = (0, 1)$ is unit in multiplication.



Theorem 2.3 (The location-scale group action)
Any action is given by a scaling and a shift.

We return to the **Example 1.1** to see if the statistics below are equivariant.

Example 2.3 (Continuation of Example 1.1) The given $t = \bar{x}$ is unbiased for $\tau = \mu$ if $\mathbb{E}(Z_i) = 0\ \forall i$. Then it is equivariant. A small excerpt of the justification is given below:

$$a + bx_i = a + b\bar{x}$$

$$t(gx) = gt(x)$$

To check whether statistic $t = \alpha\bar{x} + \beta$ is equivalent, the following consideration is made:

$$\alpha(\alpha \mp x) + \beta s(a + x) = \alpha \cdot [a + \bar{x}] + \beta s = \alpha a + \alpha\bar{x} + \beta s.$$

The statistic in this expression is equivalent if $\alpha = 1$. If $\alpha = 1$ the expression can be written as follows:

$$(b\bar{x}) + \beta s(bx) = b\bar{x} + b\beta s = \beta t(x).$$

Further on is $t = s^2$ equivariant with respect to location-scale. Likewise $t = s$ is equivariant, just like $t = \alpha s$ is also equivariant if α is positive (a little reminder: $s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$).

3 Group family

This chapter deals with group families and their definition. Furthermore, two examples of these are given.

Definition 3.1 (Group family)
$x = \theta z$ has a group familiy of distribution when $\theta \in R(\theta) = \Omega_\theta = Group$ with R as the range. Further z has a known distribution an θZ is a group action.

The following definition is a more general definition of a group family.

Definition 3.2 (Group invariant family)
Model invariance with respect to action of $G \Leftrightarrow$
$x = gz$ with $g \in G$ and group action and distribution of z is known when α is known, $\theta = (g,\ \alpha)$.

It should be mentioned that a group family is always a group invariant family.

Example 3.1 (Gamma) $X_i = \beta \cdot F_\alpha^{-1}(u_i)$ with $u_i \overset{\text{iid}}{\sim} U(0,\ 1)$ and $\theta = (\alpha,\ \beta)$.

Example 3.2 (Bernoulli) $X_1,\ \ldots,\ X_n \overset{\text{iid}}{\sim} B(p)$ and $g(x) = 1 - x$. Another way of writing is $[g(x)]_i = 1 - x_i \overset{\text{iid}}{\sim} B(1-p)$. Then this group has two elements, the identity an the inverse.

In this case $\overline{1-x} = 1 - \bar{x}$, so $\bar{x}$ is equivariant.

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