Norwegian University of Science and Technology
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TMA4295 Statistical inference

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Lecture 1 in week 35: 'Sufficiency Introduction'

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1 Probability Theory Repetition

The lecture began with some repetition of fundamental concepts and definitions of probability theory.

Definition 1.1 (Statistic) A statistic is a function of data.

Definition 1.2 (Random variable) A random point X is a function from a sample space Ω to a sample space Ω_X

For this definition to make sense we also need the definition of a sample space and events:

Definition 1.3 (Event) An event is a measurable set.

Definition 1.4 (Sample space) A sample space S is a set equipped with a family $\mathcal{F}$ of events. $\mathcal{F}$ is a non-empty collection of subsets of S, closed under complements and countable unions.

- $\emptyset \neq \mathcal{F} \subset \{A|A \subset S\}$
- $A \in \mathcal{F} \implies A^c \in \mathcal{F}$
- $A_1, A_2 ... \in \mathcal{F} \implies \bigcup_{i=1}^{n} A_i \in \mathcal{F}$

The probability of an outcome A is denoted $P_X(A) = P(X \in A) = P\{\omega | X(\omega) \in A\}$ where $\omega \in \Omega$. P_X is known when the model θ is known. In the real world this is usually not the case.

Example 1.1 (Not all sets are events!) Let X be a simple function from the sample space to some general space: $X : S \to \mathcal{X}$. We have $R(X) = \{X_1, ..., X_n\}$. A simple function gives a partition into sets $A_i, i = 1...n$, where $A_i = (X = x_i) = \{s|X(s) = x_i\} \subset S$ Then $\mathcal{F} = \{A|A = \bigcup_{i \in J} A_i, J \subset \{1...n\}\}$ is a family of events. Any non-empty proper subset of any A_i is not an event.

2 Sufficiency

Intuitively a statistic is sufficient if it contains all relevant information. The formal definition is:

Definition 2.1 (Sufficient statistic) A statistic T is sufficient if the conditional distribution $X|T = t$ is known.

Meaning it is the same for all $theta \in R(\theta)$.

The following theorem can be used to calculate sufficient statistics.

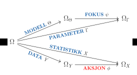
Theorem 2.1 (Factorization theorem for sufficiency) Let f be the density of the data y and let $t = t(y)$ be a statistic. The statistic t is sufficient iff $f(y \mid \theta) = g(y)h(t \mid \theta)$ where θ is the model parameter. Furthermore, g is the conditional density of the data given t, and h is the density of t.

Another theorem to identify a sufficient statistic is.

Theorem 2.2 () A statistic t is sufficient if $\frac{f_X(X|\theta)}{f_T(t|\theta)}$ does not depend on θ.

Example 2.1 (Sufficiency) Let $X_1 ... X_n$ have distribution $f(x) = \prod_{i=1}^{n} \frac{(0<X_i<\theta)}{\theta}$ then $f(x) = \theta^{-n} \cdot (0 < X_{(n)} < \theta) = g(x) \cdot h(t|\theta), t = X_{(n)}$

This means the order statistic $X_{(n)}$ is sufficient.



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