TMA4295 Statistical inference
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Exercise set 2

Read the questions carefully and make your own assumptions if needed.

1 Casella-Berger

⑥ 1, 7, 8, 9

2 Random vectors

Let X and Y be two random vectors in a finite dimensional vector space V.

a) Prove that $X + Y$ is a random vector.

b) What if V is not finite dimensional?

3 Scale Uniform distribution

Let $x_1, \ldots, x_n$ be a random sample from $\mathrm{U}(0, \theta)$ (OPEN INTERVAL!).

a) What is a natural choice for the model space Ω_Θ, and the range $\mathrm{R}(\Theta)$ of allowed models? Is it necessary that $\mathrm{R}(\Theta) = \Omega_\Theta$?

b) Explain that the assumption defines a statistical model I and specify every ingredient in the definition.

c) Is the statistical model from an exponential family?

d) Show that $y = ax$ is a random sample from $\mathrm{U}(0, a\theta)$ (OPEN INTERVAL!). This proves that the statistical model is a group model with respect to the group of transformations on the data space $\Omega_X = \mathbb{R}^n$ given by multiplication by a positive constant a. Explain!

e) Plot the likelihood function L_I of model I.

f) Is it true that the MLE does not exist in this case?

g) Explain that the graph of the likelihood function is a statistic.

h) Find a random variable S that is a minimal sufficient statistic. Hint: Any statistic that is in one-one correspondence with the likelihood is a minimal sufficient statistic.

i) Use the factorisation theorem to prove that both x and s are sufficient statistics.

j) Consider the statistical model where the observed data is $s = S(\omega)$. Calculate the distribution P_S of S. Explain how this defines a statistical model II.

k) Illustrate the level sets of $s = s(x)$ in the case $n = 2$. How are the level sets related to the sufficiency principle in this case? Explain that the level sets define a partition and then also an equivalence relation.

l) Calculate the likelihood function L_{II} of the statistical model II. Explain in what sense $L_I = L_{II}$ is true using the language of equivalence relations.

m) Determine the conditional distribution P_X^s, and illustrate in the case $s = 1$ and $n = 2$.

n) Determine $\mathrm{P}_{X_1}^s$.

o) Calculate $t = \mathrm{E}(X_1 \,|\, S = s)$.

p) Is t an unbiased estimate of $\mu = \mathrm{E}(X_1)$?

q) Is t an equivariant estimate of μ?

4 Location-Scale Uniform distribution

Let $x_1, \ldots, x_n$ be a random sample from $\mathrm{U}(\theta_1, \theta_2)$ (OPEN INTERVAL!).

a) Show that the vector statistic $a = [(x_i - \overline{x})/\hat{\sigma}]$ is ancillary. Can the Basu theorem be used to conclude that a is independent of the minimal sufficient statistic $(x_{(1)}, x_{(n)})$?

b) Reconsider all questions in 3 with suitable adjustments. Hint: It may be helpful to use the coordinates $\mu = (\theta_1 + \theta_2)/2$ and $\beta = (\theta_1 - \theta_2)/2$ in the model space.

5 General Scaled Uniform distribution

Let $x_1, \ldots, x_n$ be a random sample from $\mathrm{U}(\theta_1, \theta_2)$ (OPEN INTERVAL!). Assume that θ is restricted to be on the curve $\theta(\beta) = \beta(1 - k, 1 + k)$ where the design parameter $0 < k < 1$ is known and $\beta > 0$ is unknown.

a) Illustrate the set $\mathrm{R}(\Theta)$ of allowed θ values in the θ plane.

b) Show that the statistic $a = x_{(1)}/x_{(2)}$ is ancillary. Can the Basu theorem be used to conclude that a is independent of the minimal sufficient statistic $(x_{(1)}, x_{(n)})$?

c) Reconsider all questions in 3 with suitable adjustments.

d) Assume that the restriction by the given curve is replaced by the restriction $g(\theta) = 0$. How would you determine the MLE in this case? Consider the special cases $g(\theta) = \overline{\theta} = (\theta_1 + \theta_2)/2$ and $g(\theta) = \tilde{\theta} = \sqrt{\theta_1 \theta_2}$.

6 Casella-Berger

⑥ 10,12,15,23,29,33,34,37