Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4295 Statistical inference
Fall 2023

Exercise set 12

Ex12: Read the questions carefully and make your own assumptions if needed.

1 Equivariant Bernoulli estimators

Let the data be $X \sim \mathrm{B}(n, \theta)$ where the model θ is unknown.

a) Let $G = \{e, g_1\}$ with $e(x) = x$ and $g_1(x) = n - x$. Is G a transformation group on the range $\mathrm{R}(X)$ of the data? Is G a transformation group on $\Omega_X = \mathbb{R}$?

b) Show that the statistical model is invariant under the action of G. Are θ, $\eta = \ln(\theta) - \ln(1-\theta)$, and $\sigma = \operatorname{Std} X$ equivariant parameters? Identify the corresponding group actions. Is the parameter $1/\theta$ equivariant? Are the maximum likelihood estimators equivariant?

c) Let $p(\theta, x) = \mathrm{P}(X \geq x)$. Show that $p_0(x) = p(\theta_0, x)$ is a p-value statistic for $H_0 : \theta \leq \theta_0$. Show that $\hat{\theta}(x) = \{\theta \,|\, p(\theta, x) > \alpha\}$ is a $1 - \alpha$ level confidence set. Is it an interval estimator? Is it equivariant?

d) Let $p(\theta_0, x) = 2\min\{\mathrm{P}_0(X \geq x), \mathrm{P}_0(X \leq x), 1/2\}$. Show that $\hat{\theta}_{CP}(x) = \{\theta \,|\, p(\theta, x) > \alpha\}$ is a $1 - \alpha$ level confidence set. Is it an interval estimator? Is it equivariant? Hint: Start by proving $p(g\theta, gx) = p(\theta, x)$.

e) Let $p(\theta_0, x) = \mathrm{P}_0(\beta(\theta_0, X) \leq \beta(\theta_0, x))$ where $\beta(\theta_0, x) = \min\{\mathrm{P}_0(X \geq x), \mathrm{P}_0(X \leq x), 1/2\}$. Show that $\hat{\theta}_B(x) = \{\theta \,|\, p(\theta, x) > \alpha\}$ is a $1 - \alpha$ level confidence set. Is it an interval estimator? Is it equivariant? Prove $\hat{\theta}_B \subset \hat{\theta}_{CP}$. Hint: $q_0(x) = \mathrm{P}_0(p_0(X) \leq p_0(x)) \leq p_0(x)$ if p_0 is a p-value statistic for H_0.

f) Let $p(\theta_0, x) = \mathrm{P}_0(\lambda(\theta_0, X) \leq \lambda(\theta_0, x))$ where $\lambda(\theta_0, x) = f(x, \theta_0)/f(x, x/n)$ and $X \sim f$. Show that $\hat{\theta}_L(x) = \{\theta \,|\, p(\theta, x) > \alpha\}$ is a $1 - \alpha$ level confidence set. Is it an interval estimator? Is it equivariant?

2 p-value theory

a) Let p be a p-value statistic for H_0. Prove that $q(x) = \sup_{H_0} \mathrm{P}(p(X) \leq p(x))$ is a p-value statistic. Can the test $(p \leq \alpha)$ be better than the test $(q \leq \alpha)$?

b) Let $p(\tau_0, x)$ be a p-value statistic for $H_0 : \tau = \tau_0$ and let $q(\tau_0, x) = \sup_{H_0} \mathrm{P}(p(\tau_0, X) \leq p(\tau_0, x))$. Show that both p and q are p-value functions. Can the confidence set $(p > \alpha)$ be better than the confidence set $(q > \alpha)$?

c) Let $p(\tau, x) = p(g\tau, gx)$ for a group action and a p-value function p for τ. Is $\{\tau \,|\, p(\tau) > \alpha\}$ an equivariant $(1 - \alpha)$ level confidence set?

3 Convergence

Let $x_n = 1/n$ for $n \in \mathbb{N}$, $\mu_n(dx) = \delta_{x_n}(dx)$, and $X_n \sim \mu_n$.

a) Prove that $\lim x_n = 0$.

b) Does $X_n \to 0$ in distribution? What about $\mu_n(A)$? What about $\phi(X_n)$?

c) Does $X_n \to 0$ in probability?

d) Does $X_n \to 0$ in $L^2(\Omega)$?

4 Asymptotic theory

Let $X_1, \ldots, X_n$ be a random sample from $\mathrm{G}(\alpha, \beta)$ and let $Y_n = \overline{X}$, $Z_n = (X_1 \cdots X_n)^{1/n}$, $S_n^2 = [(X_1 - Y_n)^2 + \cdots + (X_n - Y_n)^2]/(n-1)$, and $W_n = \mathrm{E}(S_n \,|\, Y_n, Z_n)$.

a) Is Y_n a consistent estimator of $\mathrm{E}(X_1)$? Discuss the claim: $Y_n \to \alpha\beta$.

b) Is S_n a consistent estimator of $\mathrm{Std}(X_1)$? What about W_n?

c) Let $\lambda_0(x)$ be the likelihood of $H_0 : \beta = \beta_0$. Use Wilk's theorem to determine the asymptotic distribution of $\lambda_0(X)$. How can an approximate confidence interval for β be determined from this?

d) Is S_n an efficient estimator? Is W_n an efficient estimator?

5 Casella-Berger

(10) 1, 3, 7, 17, 31, 41, 44, 45, 48