Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4295 Statistical inference
Fall 2023

Exercise set 1

1 Casella-Berger

① 1, 11, 12, 35, 49 ② 10, 27

2 Random variables

a) Define the cumulative distribution function F of a random variable X.

b) Let f be a density. Explain what $X \sim f$ means.

c) Explain what $X \sim \mathrm{N}(\mu, \sigma^2)$ means.

d) Prove $\mathrm{E}\,\phi(X) = \mathrm{E}_X\,\phi$ for a simple ϕ.

e) Prove that the set of random variables is a vector space.

3 Random vectors

Can you generalize and solve the problems in 2 for the case of random vectors?

4 Conditional distributions in the plane

Let $S = U_1 - U_2$ and $T = U_1/U_2$ where U_1, U_2 is a random sample from $\mathrm{U}(0, 1)$.

a) Illustrate $a = (u_1 \leq \frac{1}{2})$, $(u_1/u_2 = 1)$, and $(u_1 - u_2 = 0)$ in the u_1u_2-plane.

b) Prove that $(S = 0) = (T = 1)$.

c) Calculate $\mathrm{P}(A \,|\, S = 0)$ and $\mathrm{P}(A \,|\, T = 1)$. Is the result a paradox?

5 Level sets and disintegration in the plane.

Let $\mu(dx) = dx_1 dx_2$ be Lebesgue measure in the plane. Define $\boldsymbol{s}(x_1, x_2) = x_1 - x_2$ and $\boldsymbol{t}(x_1, x_2) = x_1/x_2$. Let $\tilde{\mu}_{\boldsymbol{s}}(ds) = ds$ and $\tilde{\mu}_{\boldsymbol{t}}(dt) = dt$.

a) Illustrate the level sets of $\boldsymbol{s}$ and $\boldsymbol{t}$. Do they give partitions of the plane?

b) Show that $\tilde{\mu}_{\boldsymbol{s}}$ and $\tilde{\mu}_{\boldsymbol{t}}$ are pseudo-distributions for $\boldsymbol{s}$ and $\boldsymbol{t}$.

c) Find disintegrations $\mu^s(dx)\tilde{\mu}_{\boldsymbol{s}}(ds)$ and $\mu^t(dx)\tilde{\mu}_{\boldsymbol{t}}(dt)$ of μ.

d) Find $\mathrm{P}_X(dx \,|\, \boldsymbol{s}(X) = s)$ and $\mathrm{P}_X(dx \,|\, \boldsymbol{t}(X) = s)$ where $X \sim f(x)\,\mu(dx)$.

6 Gamma distribution

Let the data $x_1, \dots, x_n$ be a random sample from $\mathrm{G}(\alpha, \beta)$.

a) Explain that this defines a statistical model.

b) Determine the canonical parameter and statistic for this exponential family.

c) Is the statistical model a group model?

d) Find a data generating model for the data and for the canonical statistic.

e) Find a pivotal based on the canonical statistic.

f) Is the canonical statistic minimal sufficient? Is it complete?

7 Casella-Berger

(6) 1, 7, 8, 9

Read the questions carefully and make your own assumptions if needed.