EXAM 9/12-22. TMA4295 Stat.inf.. Solution sketch.

a) $EX = \mu = t/\theta = x$

$$\hat{\theta}_1 = \frac{t}{\underline{\underline{x}}} = \frac{10.5}{5} = \cancel{2.5}\ \underline{\underline{2.05}}$$

$$L_2 = \prod_{i=1}^{n} \theta^{-1} \cdot e^{-y_i/\theta} = \theta^{-n} \cdot e^{-n\bar{y}/\theta} \qquad n=5$$

$$= \left[(\bar{y}/\theta)\, e^{-\bar{y}/\theta} \right]^{n} \cdot \bar{y}^{-n}$$

$$1 = \bar{y}/\hat{\theta}_2 \ ): \ \hat{\theta}_2 = \underline{\underline{\bar{y}}} = \frac{10.5}{5} = \cancel{2.5}\ \underline{\underline{2{,}05}}$$

$$L_3 = \prod_{i=1}^{m} \theta \cdot e^{-w_i\theta} = \left[\bar{w}\theta\, e^{-\bar{w}\theta} \right]^{m} \cdot \bar{w}^{-m}$$

$$\pi(\theta | w) \propto \theta^{m-1} \cdot e^{-m\bar{w}\theta} \qquad m=3$$

$$\theta | w \sim G(m, (m\bar{w})^{-1})$$

$$\hat{\theta}_3 = \underline{\underline{\bar{w}^{-1}}} = 2{,}5\,\underline{\underline{5}}$$

a) Common mistakes

(i) Bayes ?

(ii) Only 1 obs. used in E3 ?

b) $E\, tX^{-1} = t \cdot \sum_{x=0}^{\infty} \frac{1}{x} \cdot \frac{\mu^x}{x!} e^{-\mu} = \infty$

$\neq \theta \Rightarrow \hat{\theta}_1$ is not unbiased

$E\,\bar{Y} = E\,Y_1 = \theta \Rightarrow \hat{\theta}_2$ is unbiased

$W_i \sim \mathrm{Exp}(\theta^{-1})$

$\bar{W} \sim \frac{1}{m} G(m, \theta^{-1}) \sim G(\alpha, \beta)$

$\alpha = m, \beta = (m\theta)^{-1}$

$E \frac{1}{\bar{W}} = \frac{1}{\Gamma(\alpha)} \cdot \int_0^{\infty} \frac{1}{x} \cdot x^{\alpha-1} \cdot e^{-x} dx \cdot (m\theta)$

$= \frac{\Gamma(\alpha-1)}{\Gamma(\alpha)} \cdot \frac{1}{\Gamma(\alpha-1)} \int_0^{\infty} x^{(\alpha-1)-1} \cdot e^{-x} dx \cdot m\theta$

$= \theta \cdot \frac{\alpha \cdot \Gamma(\alpha-1)}{\Gamma(\alpha)} = \theta \cdot \frac{\alpha\, \Gamma(\alpha-1)}{(\alpha-1) \cdot \Gamma(\alpha-1)}$

$= \frac{m}{m-1} \theta = \frac{3}{2} \theta \neq \theta$ $\hat{\theta}_3$ not unbiased b.) (Continued) Comment:

Proper Bayes estimators are never unbiased, so it is tempting to conclude that $\hat{\theta}_3$ is biased without calculation.

Problem: $\pi(\theta) = \theta^{-1}$ is not proper.

A calculation is hence needed to prove that $\hat{\theta}_3$ is biased.

b) Common mistakes:

(i) $\int$ instead of $\sum$ for discrete.

(ii) $E\left(\frac{1}{X}\right) = \frac{1}{E(X)}$ WRONG!

(iii) Very few realized that $E\left(\frac{1}{X}\right) = \infty$.

c) $EW^2 = E(W - \mu + \mu)^2$

$= \mathrm{Var}\, W + \mu^2$

$\therefore \mathrm{Var}\, W = EW^2 - \mu^2$

The estimator $\hat{\theta}_1$ does not have a well defined standard deviation, so the standard uncertainty is not defined.

(Alternatively it can be argued that the standard uncertainty equals $+\infty$)

Note also that

$$E(\hat{\theta}_1(X) - \theta)^2 = \sum_{x=0}^{\infty} \left(\frac{t}{x} - \theta\right)^2 \cdot \frac{\mu^x}{x!} e^{-\mu} = \infty$$

c) (continued)

$$\operatorname{Var} \bar{Y}_1 = \frac{1}{n^2} \cdot n \operatorname{Var} Y_1 = \frac{1}{n} \cdot \theta^2$$

$$\operatorname{Std} \bar{Y}_1 = \frac{1}{\sqrt{n}} \cdot \theta$$

$$u_2 = \frac{1}{\sqrt{n}} \hat{\theta}_2 = \frac{1}{\sqrt{5}} 2s \approx 0{,}89443 s$$

$$\approx 0{,}9 s$$

$$\operatorname{Var} \bar{W}^{-1} = E \bar{W}^{-2} - \left(E \bar{W}^{-1}\right)^2$$

$$= \theta^2 \cdot \left[\frac{9}{2} - \frac{9}{4}\right] = \theta^2 \cdot \frac{9}{4} \Big\} = \left[\frac{m}{m-1} \frac{1}{\sqrt{m-2}} \theta\right]^2$$

$$E \bar{W}^{-2} = \frac{1}{\Gamma(\alpha)} \cdot \int_0^\infty x^{-2} \cdot x^{\alpha-1} \cdot e^{-x} \, dx \cdot (\theta \alpha)^2$$

$$= \frac{\Gamma(\alpha-2)}{\Gamma(\alpha)} \cdot (\theta \alpha)^2 = \frac{(\theta 3)^2}{2}, \quad \alpha = m, \quad \beta = (m\theta)^{-1}$$

$$\operatorname{Std} \bar{W}^{-1} = \frac{3}{2} \theta, \quad u_3 = \frac{3}{2} \cdot 2{,}5 s = 3{,}75 s$$

$$\approx 3{,}8 s$$

): Use $\hat{\theta}_2$ (unbiased, smaller var.)

c) (Continued) Remark:

A ISO GUM report:

$$\theta = 2.0(?)\,s \qquad \text{(Exp. 1)}$$

$$\theta = 2.0(9)\,s \qquad \text{(Exp. 2)}$$

$$\theta = 2.5(3.8)\,s \qquad \text{(Exp. 3)}$$

where the number in parenthesis gives the standard uncertainty.
Note: In exp. 3 the standard uncertainty is larger than the estimate. Due to small sample size the estimator is strongly skewed, and the STD is not a good measure of accuracy. Better to estimate distrib. of estimator:

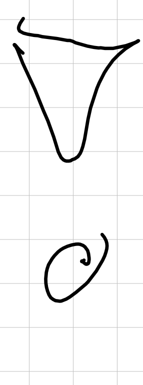
$$\hat{\theta}_1 \sim \frac{t}{\text{Poisson}(t/\hat{\theta}_1)}, \quad \hat{\theta}_2 \sim G\left(n, \frac{\hat{\theta}_2}{n}\right), \quad \hat{\theta}_3 \sim IG(\cdot)$$

c) (Continued)

It is also possible to estimate using $u = s$. This is not optimal, but gives an estimate of the standard deviation = a standard uncertainty.

This will (typically, and in this case) give an optimistic estimate – but much better than no statement about uncertainty.

): USE THE



ISO GUM

c) (Continued)

Alternative:

$$\bar{W} \sim G(m, (m\theta)^{-1})$$

$$\bar{W}^{-1} = \hat{\theta}_3 \sim IG(m, (m\theta)^{-1})$$

p. 30 in formulas:

$$E\,\hat{\theta}_3 = m\theta \cdot \frac{1}{m-1} = \frac{m}{m-1}\,\theta$$

$$\begin{aligned} \operatorname{Var} \hat{\theta}_3 &= (m\theta)^2 \cdot (m-1)^{-2} \cdot (m-2)^{-1} \\ &= \theta^2 \cdot \frac{1}{m-2} \cdot \left(\frac{m}{m-1}\right)^2 \end{aligned}$$

$$\operatorname{Std} \hat{\theta}_3 = \theta \cdot \frac{m}{m-1} \cdot \frac{1}{\sqrt{m-2}} \qquad \text{etc.}$$

Comment: $\operatorname{Std}(\theta \mid w) = \frac{1}{\sqrt{m}} \cdot \hat{\theta}_3$ is the standard uncertainty from a Bayesian perspective.

d)

$$L_1 = \frac{1}{5!} \cdot \left(\frac{105}{\theta}\right)^5 \cdot e^{-\frac{105}{\theta}}$$

$$L_2 = \theta^{-5} \cdot e^{-\frac{105}{\theta}}$$

so the (strong) likelihood principle says that the inference should be the same. This fails here since the reported standard errors are not equal. The inference is not in harmony with the likelihood principle.

$L_3 = \theta^3 e^{-3\bar{w}\theta}$ is in one-one correspondence with $\bar{w}$, so $\bar{w}$ is a minimal sufficient statistic. The inference is in harmony with the sufficiency principle.

c) Common mistakes :

(i) Did not estimate Std.

(ii) Unable to calculate Std.

(iii) $\text{Var}\left(\frac{1}{X}\right) = \frac{1}{\text{Var } X}$ WRONG

d) (Continued)

Birnbaum's theorem says that the likelihood principle is equivalent with the sufficiency & conditionality principles. The inference in both exp. 1 & 2 are in harmony with the sufficiency principle. It follows that the inference must be in conflict also with the conditionality principle.

d) Common mistakes :

(i) Birnbaum & principles totally messed up.

(ii) E1 and E2 not compared.

e) $E \frac{1}{\overline{W}} = \theta \cdot \int_0^\infty x^{-1} \cdot x^{m-1} \cdot e^{-x} \, dx \cdot \Gamma(m)^{-1} \cdot m$

$= \theta \cdot \frac{\Gamma(m-1)}{\Gamma(m)} \cdot m$

The UMVU estimator is

$\tilde{\theta}_3 = \frac{\Gamma(m)}{m\Gamma(m-1)} \cdot \overline{W}^{-1} = \frac{m-1}{m} \overline{W}^{-1}$

$= \frac{2}{3} \cdot \hat{\theta}_3 \approx 1{,}675$

(see also answer to b.))

It is unique since it is a function of a complete sufficient statistic. This follows from the characterization of complete suff. statistics for exp. families given in the textbook.

e) Mistakes :

(i) Unable to calculate with symbols.
This is common on all 10 problems.

f)

$$g_1(w_1, w_2, w_3) = \left(\frac{1}{\theta_1} w_1, \frac{1}{\theta_1} w_2, \frac{1}{\theta_1} w_3\right)$$

defines a family of one-one transformations of $\mathbb{R}^3_+$.
It is a group:

a) $g_1 \circ g_2 = g_3$ $\quad \theta_3 = \theta_1 \cdot \theta_2$

b) $g_1^{-1} = g_2$ $\quad \theta_2 = \theta_1^{-1}$

c) $g_1 = e$ when $\theta_1 = 1$.

(The associative property holds since it is a subgroup of the (large) group of 1–1 measurable transformations.)

d) (continued)

(W_1, W_2, W_3) i.i.d.

$\exp(\theta^{-1})$

$\Rightarrow \quad g_1(W_1, W_2, W_3)$ i.i.d.

$\exp((\theta, \theta)^{-1})$

so the statistical model is a group model with respect to the given group of transformations.

Alternative: Group model for the minimal suff. data $\overline{W} \sim G(m, m/\theta)$ with $m=3$.

f) (continued)

$$\overline{W} = \frac{1}{\theta} \cdot u \quad , \quad u \sim G(m, m^{-1})$$

Equivariance :

$$\left\{ \begin{array}{l} \Gamma(m) = (m-1)\Gamma(m-1) \\ = (m-1)\cdot(m-2)\cdot\Gamma(m-2) \end{array} \right.$$

$$\check{\theta}(\overline{W}) = \cancel{\overset{\approx 0{,}835}{\frac{1}{\overline{W}}}} \cdot \check{\theta}(1) = \cancel{\frac{1}{\overline{W}} \cdot \frac{m-2}{m} = \frac{1}{\overline{W}\cdot 3}}$$

since:

$$\rho = E(\check{\theta}(\overline{W}) - \theta)^2$$

$$= \theta^2 \cdot E\left(\frac{\check{\theta}(1)}{U} - 1\right)^2$$

$$0 = \partial_{\check{\theta}} \cdot = E\left[2\cdot\left(\frac{\check{\theta}(1)}{U} - 1\right)\cdot\frac{1}{U}\right]$$

$$): \check{\theta}(1) = \frac{E\,U^{-1}}{E\,U^{-2}} = \frac{\cancel{m}}{\cancel{m-1}} \cdot \frac{\cancel{(m-1)}\cdot(m-2)}{m^{\cancel{2}}}$$

$$E\,U^{-1} = m \cdot \int_0^\infty x^{-1} \cdot x^{m-1} \cdot e^{-x}\,dx \cdot \frac{1}{\Gamma(m)}$$

$$= m \cdot \frac{\Gamma(m-1)}{\Gamma(m)} = \frac{m}{m-1} , \quad E\,U^{-2} = m^2 \cdot \frac{\Gamma(m-2)}{\Gamma(m)} = \frac{m^2}{(m-1)(m-2)}$$

f) (Continued)

Alternativ calculation: Can consider invariant loss $L = (\check{\theta} - \theta)^2/\theta^2$. Risk:

$$\rho = E\left[\left(\check{\theta} - \frac{U}{\bar{W}}\right)^2 \Big/ \frac{U^2}{\bar{W}^2}\right]$$

$$= E\left[\left(\frac{\bar{W}}{U}\cdot\check{\theta} - 1\right)^2\right]$$

$$0 = \partial_{\check{\theta}(1)} E\left[\left(\frac{\check{\theta}(1)}{U} - 1\right)\cdot\frac{1}{U}\right] \text{ etc as before.}$$

f) Mistakes :

(i) Group model ?

(ii) Very few found the best equivariant estimator.

(Unable to do calculation with symbols.)

g) Large x indicates small θ, so reject H_0 if X is large. The p-value is then:

$$p = \sup_{\theta \in H_0} P(X \geq x_0) = P(X \geq \overset{\overset{5}{\shortparallel}}{x_0} \mid \mu_* = \overset{\overset{4}{\shortparallel}}{\tfrac{10}{2,5}})$$

p-18 Poisson table.

$$= 1 - \sum_{x=0}^{4} \frac{\mu_*^x}{x!} \cdot e^{-\mu_*} = 1 - 0{,}6288$$

$\approx \underline{\underline{0{,}37}}$ not less than or equal to 0,05

): $\underline{\underline{H_0 \text{ is not rejected}}}$

$$L_1 = \frac{\mu^x}{x!} e^{-\mu} = e^{x \cdot \ln \mu - \mu} \cdot \frac{1}{x!}$$

): Yes, exponential family with natural parameter $\ln \mu = \ln \frac{t}{\theta}$
$= \ln t - \ln \theta$
(or $-\ln\theta$ by choice)

g) (Continued)

$\theta_0 < \theta_1$ implies the likelihood ratio $L(\theta_1)/L(\theta_0) = C \cdot \exp(X \cdot \ln \frac{\mu_1}{\mu_0})$ is decreasing in X, so the Karlin-Rubin gives that the test is most powerful (The level is smaller than 5%, but optimal at this smaller level.)

Small θ corresponds to short expected times between gamma particles, so H_1 corresponds to large radioactivity.

): Can be used to check if there is high radioactivity. (Possibly with different choice of threshold)

g) Remark:

Can base the test on $\hat{\theta}_1 = t/x$.
Small $\hat{\theta}_1$ indicates H_0 is wrong,
so reject H_0 if $\hat{\theta}_1$ is small.
This is equiv. with the previous,
but demonstrates construction of
a test from an estimator.

g) Mistakes :

(i) Unable to do test.

(ii) Failure to understand what a test is.

(iii) p value ?

h)

$$L = \theta^m \cdot e^{-m\bar{W}\theta}, \quad \lambda = \frac{\hat{L}_0}{\hat{L}_-} = \frac{\theta_0^m \cdot e^{-m\bar{W}\theta_0}}{\bar{W}^{-m} \cdot e^{-m}}$$

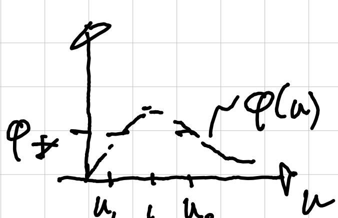
$$\lambda = \left[e \cdot (\theta_0 \bar{W}) \cdot e^{-\theta_0 \bar{W}}\right]^m$$

Let $\varphi(u) = u \cdot e^{-u}$. Observe that

$$u = \theta \bar{W} \sim G(m, m^{-1})$$

Reject if λ is small, so accept if λ is large. Choose φ_* so that

$$P(\varphi(\theta \bar{W}) \geq \varphi_*) = 1 - \alpha = 95\%$$



$\varphi(u_1) = \varphi(u_2) = \varphi_*$ Yes! equivariant.

$$u_1 \leq \theta \bar{W} \leq u_2$$

$$S = \hat{\theta}_3 \cdot [u_1, u_2]$$

$$S\left(\tfrac{1}{c}(W_1, W_2, W_3)\right) = c \cdot S(W_1, W_2, W_3)$$

h) Reformulation:

u_1 and u_2 are determined by

$$95\% = 1-\alpha = P\big(u_1 \le \underbrace{G(m, m^{-1})}_{= \theta \cdot \overline{W}} \le u_2\big)$$

$$\varphi(u_1) = \varphi(u_2) \quad (= u_i e^{-u_i})$$

CI is $\hat{\theta}_3 \cdot [u_1, u_2]$

h) Common mistakes :

(i) Unable to invert test to get interval.

(ii) Equivariant ?

i) This proves that the CDF of $\hat{\theta}_3(W_1, W_2, W_3)$ is defined, so it is relevant in experiment 3.

Axioms for events are
(i) $\emptyset$ is an event (ii) The complement of an event is an event (iii) The countable union of events is an event. It follows that countable intersections of events is also an event.

A) The sum of two random variables is a random variable:

$$(X+Y>z) = \bigcup_{q\in\mathbb{Q}} \{(X>q)\cap[q>z-Y]\}$$

is an event, so $(X+Y\leq z)$ is an event. Note: $[a<b]$ iff $[\exists c \;\; a<c<b]$ iff $[\exists q\in\mathbb{Q} \;\; a<q<b]$ because any real c is the limit of a sequence of rational numbers.

i) (continued)

B) $\frac{1}{m} X$ is a random variable if X is : $\left(\frac{1}{m} X \leq z\right) = (X \leq mz)$

C) X^{-1} is a random variable if X (>0) is :

$$(X^{-1} \leq z) = (X \geq z^{-1})$$

$$= \left[(X < z^{-1})\right]^c$$

$$(X < q) = \bigcup_{n=1}^{\infty} \left(X \leq q - \frac{1}{n}\right)$$

Taken together this proves that $\overline{W}^{-1}$ is a random variable

so $(\overline{W}^{-1} \leq q)$

is an event for any q. //

i) (Continued)

Alternative proof:

$(w_1, w_2, w_3) \mapsto \bar{w}^{-1}$

is measurable since it is continuous.

Filling in the details for obtaining a full proof starting from the axioms and concluding as above is a longer path. (but rewarding)

Remark from student: (Good answer)

"In E3 we look at the estimator $\hat{\theta}_3 = 1/\bar{w}$, so it's good to know that $\hat{\theta}_3 \le q$ indeed is an event."

i) Mistakes

i) Missing use of rules for events $\equiv$ No precise understanding of the concept of an event.

ii) Some claim that

$$(W_1 + W_2 \leq w_1 + w_2)$$

$$\Leftrightarrow (W_1 \leq w_1) \cap (W_2 \leq w_2)$$

This is WRONG.

iii) Many similar wrong claims.

j) $L_2 = \theta^{-n} \cdot e^{-n\bar{y}/\theta}$, $L_3 = \theta^{m} \cdot e^{-m\bar{w}\theta}$

$$\ln(L_2 L_3) = (m-n)\ln\theta - [n\bar{y}\theta^{-1} + m\bar{w}\theta]$$

$$0 = \partial_\theta \ln L = (m-n)\cdot\theta^{-1} + n\bar{y}\theta^{-2} - m\bar{w}$$

$$\theta^{-1} = \frac{(n-m) \pm \sqrt{(n-m)^2 + 4n\bar{y}m\bar{w}}}{2n\bar{y}}$$

Positive solution

$$\hat{\theta} = \bar{y} \cdot \left[\frac{\sqrt{(n-m)^2 + 4nm\bar{y}\bar{w}} + (n-m)}{2n}\right]^{-1}$$

$$\approx 2.0s \cdot \left[\frac{\sqrt{4 + 4\cdot 15 \cdot \frac{2.0}{2.5}} + 2}{10}\right]^{-1}$$

$$\left.\approx 2.17s\right\} = \hat{\theta}_2 \cdot \left[\frac{\sqrt{(n-m)^2 + 4nm\hat{\theta}_2/\hat{\theta}_3} + (n-m)}{2n}\right]^{-1}$$

$$\left(n = m \text{ case}: \hat{\theta} = \bar{y}^{1/2} \cdot \bar{w}^{-1/2} = \sqrt{\hat{\theta}_2 \hat{\theta}_3}\right)$$

j) (Continued)

$\hat{\theta}_2 = \theta \cdot u$, $u \sim G(n, n^{-1})$

$\hat{\theta}_3 = \theta \cdot v^{-1}$, $v \sim G(m, m^{-1})$

$a = uv$ known distribution

$(\hat{\theta}_2, \hat{\theta}_3)$ is minimal since in one-one correspondence with the likelihood.

It is not complete since

$E(\hat{\theta}_2(\underline{Y}) - \hat{\theta}_3(\underline{W})) = 0$

but $\hat{\theta}_2 - \hat{\theta}_3 = \bar{y} - \bar{w}^{-1} \neq 0$

j) (Continued) Inference can be based on $\hat{\theta}_2, \hat{\theta}_3$ and is then a scale model. Equivalently based on $(\hat{\theta}_3, a)$, and then conditionally given a:

$$\hat{\theta}_3 = \theta \cdot V^a, \quad V^a \sim V \mid a$$

Calculation as in f) gives:

$$\hat{\theta} = \hat{\theta}_3 \cdot \frac{E(V^{-1} \mid A = a)}{E(V^{-2} \mid A = a)}$$

Interval $S = \hat{\theta}_3 \cdot [s_1, s_2]$ and loss $L = \hat{\theta}_3 \cdot (s_2 - s_1)\theta^{-1} + k \cdot (\theta \notin S)$:

$$\rho = (s_2 - s_1) \cdot E V^a + k \cdot E(1 \notin V^a \cdot [s_1, s_2])$$

Calculus gives s_1 & s_2. Suitable choice of k gives level $1-\alpha$.

j.) Mistakes

a) Very few calculated the MLE.

b) Failure to argument for 'a' being ancillary.

c) Very few understand concept of 'equivariance'.