Solution to exercise 5.5

The exercise asks us to show, for $X_1, \dots, X_n$ iid random variables, that

$$f_{\bar{X}}(x) = n f_{X_1+\cdots+X_n}(nx),$$

even if the moment-generating function does not exist.

First of all, define $Y = X_1 + \cdots + X_n$ and note that $\bar{X} = \frac{1}{n}Y$.

We now have to show that

$$f_{\bar{X}}(x) = n f_Y(nx),$$

to complete the exercise.

We saw that the random variable $\bar{X}$ is a transformation of the random variable Y, i.e $\bar{X} = g(Y) = \frac{1}{n}Y$. To find the pdf of $\bar{X}$, we use the formula for transformation of variables (see Theorem 2.1.5 in Casella and Berger). Thus,

$$f_{\bar{X}}(x) = f_Y(g^{-1}(x)) \left| \frac{d}{dx} g^{-1}(x) \right|$$

Here $\bar{x} = g(y) = \frac{1}{n}y$, and so $y = g^{-1}(\bar{x}) = n\bar{x}$. Evaluated in some point x we have that $f_Y(g^{-1}(x)) = f_Y(nx)$ and $\frac{d}{dx}g^{-1}(x) = n$. Then

$$f_{\bar{X}}(x) = f_Y(nx)|n| = n f_Y(nx) = n f_{X_1+\cdots+X_n}(nx).$$