TMA4265 Stochastic Processes
Week 35 – Solutions

Problem 2.71: Sum of exponentials are gamma distributed

We need to show that if $X_1, X_2, \ldots, X_N$ are independent random variables from exponential distributions with the same rate parameter λ, then their sum has a gamma distribution. One way to solve this problem is to use moment generating functions. Another way is to combine induction with the fact that if Y and Z are two independent, continuous random variables, then the sum has a probability distribution given by the convolution of the probability distributions

$$f_{Y+Z}(u) = \int_{\mathbb{R}} f_Y(u-t) f_Z(t)\,\mathrm{d}t.$$

Let us start by assuming that $S_i = X_1 + \ldots + X_i$ is distributed as

$$S_i \sim \text{Gamma}(i, \lambda), \text{i.e. } f_{S_i}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{i-1}}{(i-1)!}$$

and note that an $\text{Exp}(\lambda)$ distribution is also a $\text{Gamma}(1, \lambda)$, that is

$$X_{i+1} \sim \text{Gamma}(1, \lambda).$$

Then $S_{i+1} = S_i + X_{i+1}$ is the sum of two independent random variables and the probability distribution is

$$\begin{aligned} f_{S_{i+1}}(u) &= \int_0^\infty f_{X_{i+1}}(u-t) f_{S_i}(t)\,\mathrm{d}t \\ &= \int_0^u \lambda e^{-\lambda(u-t)} \lambda e^{-\lambda t} \frac{(\lambda t)^{i-1}}{(i-1)!}\,\mathrm{d}t \\ &= \lambda e^{-\lambda u} \int_0^u \frac{(\lambda t)^{i-1}}{(i-1)!}\,\mathrm{d}t \\ &= \lambda e^{-\lambda u} \frac{(\lambda u)^i}{i!} \end{aligned}$$

Thus $S_{i+1} \sim \text{Gamma}(i+1, \lambda)$, and it follows from mathematical induction that $X_1 + \ldots + X_N \sim \text{Gamma}(N, \lambda)$ for any $N \geq 1$.

Problem 2: Changes in stock prices

The model in the problem can be written as

$$Z = X_0 + \sum_{i=1}^{N} X_i,$$

where X_0 is always included and one or more X_i, for $i \geq 1$, may be included. The desired variance can be calculated via the law of total variance,

$$\begin{aligned} \text{Var}[Z] &= \text{E}[\text{Var}[Z|N]] + \text{Var}[\text{E}[Z|N]] \\ &= \text{E}[\sigma^2 + N\sigma^2] + \text{Var}[0 + N \cdot 0] \\ &= \sigma^2 + \nu\sigma^2 = (1+\nu)\sigma^2. \end{aligned}$$

Problem 3: Joint distribution

1.

$$\begin{aligned}
p_X(x) &= \sum_y p(x,y) = \sum_y \exp(-2\lambda)\frac{\lambda^{x+y}}{x!\,y!} \\
&= \exp(-\lambda)\frac{\lambda^x}{x!} \underbrace{\sum_y \exp(-\lambda)\frac{\lambda^y}{y!}}_{1} \\
&= \exp(-\lambda)\frac{\lambda^x}{x!}
\end{aligned}$$

Hence, X is Poisson distributed with parameter λ, e.g. $X \sim \mathcal{P}(\lambda)$. Analogously, $Y \sim \mathcal{P}(\lambda)$. We find that X and Y are independent, since $p(x,y) = p_X(x)p_Y(y)$ is fulfilled. Hence,

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) \overset{X \text{ and } Y \text{ indep.}}{=} E(X)E(Y) - E(X)E(Y) = 0,$$

$\Rightarrow$ the covariance of X and Y is zero.

2. This is analogous to the proof in the lecture where we used that $X + Y \sim P(2\lambda)$. Hence

$$\begin{aligned}
P(X|Z = X+Y) &= \frac{P(X = x, Z = x+Y)}{P(Z = z)} \\
&= \frac{P(X = x, Z = x+Y)}{P(Z = z)} \\
&= \frac{P(X = x, Y = z-x)}{P(Z = z)} \\
&= \frac{P(X = x)P(Y = z-x)}{P(Z = z)} \\
&= \frac{\exp(-\lambda)\frac{\lambda^x}{x!}\exp(-\lambda)\frac{\lambda^{z-x}}{(z-x)!}}{\exp(-2\lambda)\frac{(2\lambda)^z}{z!}} \\
&= \frac{\frac{\lambda^z}{x!(x-z)!}}{\frac{(2\lambda)^z}{z!}} \\
&= \frac{z!}{x!(x-z)!}\left(\frac{1}{2}\right)^z \\
&= \binom{z}{x}\left(\frac{1}{2}\right)^z \\
&= \binom{z}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{z-x}
\end{aligned}$$

Hence, $X|X+Y$ is binomially distributed with size z and probability 0.5, e.g. $X|X+Y \sim \mathcal{B}(z, 0.5)$.

3.

$$\begin{aligned}
\mathrm{Cov}(X+Y, X-Y) &= E((X+Y)(X-Y)) - E(X+Y)E(X-Y) \\
&= E(X^2 - Y^2) - (E(X) + E(Y)) \cdot \underbrace{(E(X) - E(Y))}_{0} \\
&= E(X^2) - E(Y^2) \\
&= 0
\end{aligned}$$

Problem 4: Expectation

1. This exercise is a special case of Example 3.15 in the book. Let N_2 be the number of necessary rolls until two consecutive sixes appear, and let M_2 denote its mean. We condition on N_1 the number of trials needed for one six. Hence

$$M_2 = E(N_2) = E(E(N_2|N_1)),$$

where

$$\begin{aligned} E(N_2|N_1) &= \underbrace{p(N_1+1)}_{\text{case 1}} + \underbrace{(1-p)(N_1+1+E(N_2))}_{\text{case 2}} \\ &= p \cdot N_1 + p + N_1 + 1 - p \cdot N_1 - p + (1-p)E(N_2) \\ &= N_1 + 1 + (1-p)E(N_2) \end{aligned}$$

It takes N_1 rolls to get one six, then either the next roll is a six (with probability $p = \frac{1}{6}$)) as well, and we are done (case 1), or it is not a six (with probability $1 - p = \frac{5}{6}$) and we must begin anew (case 2). For case 2, it is important to have in mind that we have already needed $N_1 + 1$ rolls to get that far.

Taking expectations of both sides of the preceding yields

$$M_2 = M_1 + 1 + (1-p)M_2$$

or

$$M_2 = \frac{M_1 + 1}{p}.$$

Since N_1, the time of the first six, is geometric with parameter p we see that

$$M_1 = \frac{1}{p} = \frac{1}{\frac{1}{6}} = 6,$$

and thus

$$M_2 = \frac{6+1}{\frac{1}{6}} = 42.$$

The expected of rolls we need until the firs pair of consecutive sixes appears is 42.