

TMA4265 Stochastic Processes

Week 35 – Solutions

Problem 1.12: Probability of being first event to occur

The experiment consists of repeated trials where in each trial E , F or neither occurs, i.e. the outcome is selected from $\{E, F, (E \cup F)^c\}$. In each repetition the probability of each of these events is the same.

For the experiment to consist of n repetitions and ending with E it is necessary that $(E \cup F)^c$ occurs $n - 1$ times and that E occurs the last time. Let N denote the random variable describing the total number of repetitions. We may then write

$$P(E \cap (N = n)) = P(E)(1 - P(E) - P(F))^{n-1}.$$

The desired probability is then found by summing over all possible values of n ,

$$\begin{aligned} P(E \text{ occurs before } F) &= \sum_{n=1}^{\infty} P(E)(1 - P(E) - P(F))^{n-1} \\ &= \frac{P(E)}{P(E) + P(F)} \sum_{n=1}^{\infty} (1 - P(E) - P(F))^{n-1} (P(E) + P(F)) \\ &= \frac{P(E)}{P(E) + P(F)}, \end{aligned}$$

where the last equality is done by recognizing the sum of all probabilities in a geometric distribution.

Problem 1.13: The dice game craps

Let F be a random variable denoting the sum achieved on the first throw. There are three situations based on the value of the first throw.

Case 1. If $F = 7$ or $F = 11$ the player wins immediately.

Case 2. If $F = 2$, $F = 3$ or $F = 12$ the player loses immediately.

Case 3. If F does not take one of the aforementioned values, then she continues throwing until she either throws the sum F again, and wins, or she throws a seven, and loses. This is exactly the situation in Problem 1.12. Thus we must calculate the probability of throwing a specific sum. It can be shown that

$$P(F = i) = \begin{cases} \frac{i-1}{36}, & i = 2, \dots, 7, \\ \frac{13-i}{36}, & i = 8, \dots, 12, \end{cases}$$

by counting the number of ways to achieve each sum.

We find

$$\begin{aligned} P(\text{Win}) &= P(F = 7) + P(F = 11) + \sum_{i \in \{4, 5, 6, 8, 9, 10\}} P(F = i) \frac{P(F = i)}{P(F = i) + P(F = 7)} \\ &= \frac{6}{36} + \frac{2}{36} + \frac{\left(\frac{3}{36}\right)^2}{\frac{3}{36} + \frac{6}{36}} + \frac{\left(\frac{4}{36}\right)^2}{\frac{4}{36} + \frac{6}{36}} + \frac{\left(\frac{5}{36}\right)^2}{\frac{5}{36} + \frac{6}{36}} + \frac{\left(\frac{5}{36}\right)^2}{\frac{5}{36} + \frac{6}{36}} + \frac{\left(\frac{4}{36}\right)^2}{\frac{4}{36} + \frac{6}{36}} + \frac{\left(\frac{3}{36}\right)^2}{\frac{3}{36} + \frac{6}{36}} \\ &= 0.493 \end{aligned}$$

Problem 1.44: Draws from an urn

Let U be a random variable denoting which urn is chosen, with value 1 if Urn 1 is chosen and with value 2 if Urn 2 is chosen. Since the urn is determined by a fair coin we have $P(U = 1) = P(U = 2) = 0.5$.

The question in the book can be rephrased as what is the probability that Urn 2 was selected given that a white ball was drawn. By Bayes's formula we find

$$\begin{aligned}
 P(U = 2 | \text{White ball drawn}) &= \frac{P(U = 2 \cap \text{White ball drawn})}{P(\text{White ball drawn})} \\
 &= \frac{P(\text{White ball drawn} | U = 2)P(U = 2)}{P(\text{White ball drawn} | U = 1)P(U = 1) + P(\text{White ball drawn} | U = 2)P(U = 2)} \\
 &= \frac{\frac{3}{3+12} \cdot \frac{1}{2}}{\frac{5}{5+7} \cdot \frac{1}{2} + \frac{3}{3+12} \cdot \frac{1}{2}} \\
 &= \frac{24}{74}.
 \end{aligned}$$

Problem 2.43: Expected number of draws

1. X denotes the total number of red balls removed before the first black ball is chosen and X_i is an indicator variable for whether red ball i is removed before the first black ball is chosen. Therefore, X can be expressed as

$$X = X_1 + X_2 + \dots + X_n.$$

2. We are first interested in $E[X_i] = P(X_i = 1)$. First, make the observation that the other red balls are irrelevant in calculating this probability. It does not matter whether they are drawn before or after red ball i . Second, ignoring the other red balls it is equally likely to draw red ball i or one of the black balls. We get

$$E[X_i] = P(X_i = 1) = \frac{1}{m + 1}.$$

Since the expectation of a sum is the sum of the expectations, we find

$$E[X] = E[X_1] + \dots + E[X_n] = \frac{n}{m + 1}.$$