TMA 4265 Stochastic Processes

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Solution - Exercise 1

Exercises from the text book

3.3

We have that

$$\begin{aligned}
E[X \mid Y=1] &= \sum_{k=1}^{3} k\, P(X=k|Y=1) \\
&= \sum_{k=1}^{3} \frac{k\, P(X=k, Y=1)}{P(Y=1)} = \frac{1}{P(Y=1)} \sum_{k=1}^{3} k\, p(k,1) \\
&= \frac{\sum_{k=1}^{3} k\, p(k,1)}{\sum_{k=1}^{3} p(k,1)} = \frac{1\cdot\frac{1}{9} + 2\cdot\frac{1}{3} + 3\cdot\frac{1}{9}}{\frac{1}{9}+\frac{1}{3}+\frac{1}{9}} = \underline{\underline{2}} \\
E[X \mid Y=2] &= \underline{\underline{\frac{5}{3}}} \\
E[X \mid Y=3] &= \underline{\underline{\frac{12}{5}}}
\end{aligned}$$

3.8

- X - number of rolls necessary to obtain a six
- Y - number of rolls necessary to obtain a five
- We have:

$$\begin{aligned}
P(X=1) &= \frac{1}{6} = p \\
P(X=2) &= p(1-p) \\
P(X=3) &= p(1-p)^2 \\
&\vdots \\
P(X=k) &= p(1-p)^{k-1}
\end{aligned}$$

$\Rightarrow$ Geometric distribution: $E[X] = \sum_{k=1}^{\infty} kp(1-p)^{k-1} = \frac{1}{p}$

a)

$$E[X] = \frac{1}{p} = \frac{1}{\frac{1}{6}} = \underline{\underline{6}}$$

b) Intuitively: $E[X \mid Y = 1] = E[1 + X] = 1 + E[X] = \underline{\underline{7}}$
From distribution: We have

$$\begin{aligned}
P(X = 1|Y = 1) &= 0 \\
P(X = 2|Y = 1) &= \frac{1}{6} = p \\
P(X = 3|Y = 1) &= (1-p)p \\
&\vdots \\
P(X = k|Y = 1) &= (1-p)^{k-2}p;\ k = 2, 3, \ldots
\end{aligned}$$

$$\begin{aligned}
\Rightarrow E[X \mid Y = 1] &= \sum_k k\, P(X = k|Y = 1) \\
&= 1 \cdot P(X = 1|Y = 1) + \sum_{k=2}^{\infty} kP(X = k|Y = 1) \\
&= 0 + \sum_{k=2}^{\infty} k(1-p)^{k-2}p \\
&= \sum_{k=1}^{\infty} (k+1)(1-p)^{k-1}p \\
&= \sum_{k=1}^{\infty} k(1-p)^{k-1}p + \sum_{k=1}^{\infty} (1-p)^{k-1}p \\
&= E[X] + 1 \\
&= \underline{\underline{7}}
\end{aligned}$$

Alternatively

$$\begin{aligned}
E[X \mid Y=1] &= \sum_{k=2}^{\infty} k(1-p)^{k-2}p \\
&= p\sum_{k=0}^{\infty}(k+2)(1-p)^k \\
&= p\sum_{k=0}^{\infty}k(1-p)^k + 2p\sum_{k=0}^{\infty}(1-p)^k \\
&= \frac{p(1-p)}{(1-(1-p))^2} + \frac{2p}{1-(1-p)} \\
&= \frac{1-p}{p} + 2 \\
&= \underline{\underline{7}}
\end{aligned}$$

c) From distribution:

$$\begin{aligned}
P(X=1 \mid Y=5) &= \frac{1}{5} \\
P(X=2 \mid Y=5) &= \frac{4}{5}\cdot\frac{1}{5} \\
&\vdots \\
P(X=4 \mid Y=5) &= \left(\frac{4}{5}\right)^3\cdot\frac{1}{5} \\
P(X=5 \mid Y=5) &= 0 \\
P(X=6 \mid Y=5) &= \left(\frac{4}{5}\right)^4\cdot\frac{1}{6} \\
&\vdots \\
P(X=k \mid Y=5) &= \left(\frac{4}{5}\right)^4\cdot(1-p)^{k-6}\cdot p;\ k=6,7,\ldots
\end{aligned}$$

We obtain

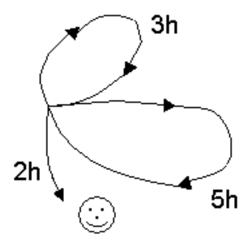
$$
\begin{aligned}
E[X \mid Y=5] &= \sum_{k=1}^{\infty} k\, P(X=k \mid Y=5) \\
&= 1 \cdot \frac{1}{5} + 2 \cdot \frac{1}{5} \cdot \frac{4}{5} + 3 \cdot \frac{1}{5} \cdot \left(\frac{4}{5}\right)^2 + 4 \cdot \frac{1}{5} \cdot \left(\frac{4}{5}\right)^3 \\
&\quad + \left(\frac{4}{5}\right)^4 \sum_{k=6}^{\infty} k(1-p)^{k-6} p \\
\sum_{k=6}^{\infty} k(1-p)^{k-6} p &= \sum_{k=1}^{\infty} (k+5)(1-p)^{k-1} p \\
&= \sum_{k=1}^{\infty} k(1-p)^{k-1} p + 5 \sum_{k=1}^{\infty} (1-p)^{k-1} p \\
&= E[X] + 5 \\
&= 11 \\
E[X \mid Y=5] &= 1,314 + \left(\frac{4}{5}\right)^4 \cdot 11 = \underline{\underline{5,82}}
\end{aligned}
$$

Alternatively

$$
\begin{aligned}
\sum_{k=6}^{\infty} k(1-p)^{k-6} p &= p \sum_{k=0}^{\infty} (k+6)(1-p)^k \\
&= p \sum_{k=0}^{\infty} k(1-p)^k + 6p \sum_{k=0}^{\infty} (1-p)^k \\
&= \frac{p(1-p)}{(1-(1-p))^2} + \frac{6p}{1-(1-p)} \\
&= \frac{1-p}{p} + 6 \\
&= 11.
\end{aligned}
$$

3.21

- N - Number of doors selected before safety
- T_i - Travel time corresponding to i'th choice
- X - Time when miner reaches safety



a)

$$\underline{\underline{X = \sum_{i=1}^{N} T_i}}$$

b)

$$\begin{aligned} P(N=1) &= \frac{1}{3} = p \\ P(N=2) &= (1-p)p \\ P(N=3) &= (1-p)^2 p \\ &\vdots \\ P(N=k) &= (1-p)^{k-1}p \end{aligned}$$

$\Rightarrow$ Geometric distribution: $E[N] = \frac{1}{p} = \frac{1}{\frac{1}{3}} = \underline{\underline{3}}$

c) – The final tour, number N, is the door to safety; $T_N \equiv 2 \Rightarrow E[T_N] = 2$

d) – We observe that the tours before number N last either three or five hours, such that

$$E[T_i \mid N = n] = \frac{1}{2} \cdot 3 + \frac{1}{2} \cdot 5 = \underline{\underline{4}} \qquad i < n$$

$$
\begin{aligned}
E[\sum_{i=1}^{N} T_i \mid N=n] &= \sum_{i=1}^{n} E[T_i \mid N=n] \\
&= \sum_{i=1}^{n-1} E[T_i \mid N=n] + E[T_N] \\
&= (n-1)\cdot 4 + 2 = \underline{\underline{4n-2}}
\end{aligned}
$$

e)

$$
\begin{aligned}
E[X] &= E[E[X \mid N]] = E[E[\sum_{i=1}^{N} T_i \mid N]] \\
&= E[4N-2] = 4\cdot E[N] - 2 = 4\cdot 3 - 2 = \underline{\underline{10}}
\end{aligned}
$$

3.24

- Flip coin with heads H and tail T, where $P(H)=p$ and $P(T)=1-p$
- N - Number of flips until at least one H and one T
- F_H - Number of flips until first H, where H is Geometric: $E[F_H]=\frac{1}{p}$
- F_T - Number of flips until first T, where T is Geometric: $E[F_T]=\frac{1}{1-p}$
- F - Outcome of first flip

a)

$$
\begin{aligned}
E[N] &= E[N \mid F=H]\cdot P(F=H) + E[N \mid F=T]\cdot P(F=T) \\
&= E[1+F_T]\cdot p + E[1+F_H]\cdot(1-p) \\
&= \left(1+\frac{1}{1-p}\right)\cdot p + \left(1+\frac{1}{p}\right)\cdot(1-p) \\
&= \underline{\underline{1+\frac{p}{1-p}+\frac{1-p}{p}}}
\end{aligned}
$$

b)
- N_H - Number of N which are H
- N_T - Number of N which are T

$$
\begin{aligned}
E[N_H] &= E[N_H \mid F=H]\cdot P(F=H) + E[N_H \mid F=T]\cdot P(F=T) \\
&= (1+E[F_T]-1)\cdot p + 1\cdot(1-p) \\
&= \frac{1}{1-p}\cdot p + (1-p) \\
&= \underline{\underline{1-p+\frac{p}{1-p}}}
\end{aligned}
$$

c) Similar as in **b)**:
$E[N_T] = \underline{\underline{p + \frac{1-p}{p}}}$
(Notice that this agrees with that $E[N_H] + E[N_T] = E[N]$)

d) M -Number of flips until at least $2H$ and $1T$

$$\begin{aligned} E[M] &= E[M \mid F = H] \cdot P(F = H) + E[M \mid F = T] \cdot P(F = T) \\ &= (1 + E[N]) \cdot p + (1 + 2 \cdot E[F_H]) \cdot (1 - p) \\ &= \left(2 + \frac{p}{1-p} + \frac{1-p}{p}\right) \cdot p + \left(1 + \frac{2}{p}\right) \cdot (1 - p) \\ &= \underline{\underline{\frac{3 + p^2}{p(p-1)}}} \end{aligned}$$

3.49

Let A be the event that A is the overall winner, and let X be the number of games. Let Y be the number of victories for A in the first two games.

$$\begin{aligned} P(A) &= P(A|Y = 0)P(Y = 0) + P(A|Y = 1)P(Y = 1) + P(A|Y = 2)P(Y = 2) \\ &= 0 + P(A)2p(1 - p) + p^2, \end{aligned}$$

such that

$$P(A) = \frac{p^2}{1 - 2p(1 - p)}$$

Similarly

$$\begin{aligned} E[X] &= E[X|Y = 0]P(Y = 0) + E[X|Y = 1]P(Y = 1) + E[X|Y = 2]P(Y = 2) \\ &= 2(1 - p)^2 + (2 + E[X])2p(1 - p) + 2p^2 \\ &= 2 + E[X]2p(1 - p), \end{aligned}$$

such that

$$E[X] = \frac{2}{1 - 2p(1 - p)}$$

We get the greatest value at $p = 0.5$:

$$E[X] = \frac{2}{1 - 0.5} = 4$$

Note that for $p = 0$ and $p = 1$ we get $E[X] = 2$, which is expected.

3.54

1. Randomly select a coin from a group of ten coins, where $n = 1, \ldots, 10$. Coin number n has probability $p = \frac{n}{10}$ of coming up heads.

2. Flip this coin until a head appears.

- N = Number of flips necessary until head appears when coin number n is flipped.

$$\begin{aligned} P(N=k) &= \sum_{n=1}^{10} P(N=k \mid \text{Coin number } n) \cdot P(\text{Coin number } n) \\ &= \sum_{n=1}^{10} \left(1-\frac{n}{10}\right)^{k-1} \cdot \frac{n}{10} \cdot \frac{1}{10} \\ &\neq (1-p)^{k-1} \cdot p \quad \Rightarrow \text{NOT Geometric distribution.} \end{aligned}$$

- For N to be a Geometric distributed stochastic variabel, all coins must have equal probability of getting heads.