Overview of TMA4265 Stochastic Processes

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Overview

Review probability

Discrete-time Markov chains

Poisson processes

Continuous-time Markov chains

Queueing theory

Brownian motion

Law of total probability

Let $B_1, B_2, \ldots$ be pairwise disjoint events with $\mathrm{P}(\bigcup_{i=1}^{\infty} B_i) = 1$

1. $\mathrm{P}(A \cap B) = \mathrm{P}(A|B)\,\mathrm{P}(B)$
2. $\mathrm{P}(A) = \sum_{i=1}^{\infty} \mathrm{P}(A|B_i)\,\mathrm{P}(B_i)$
3. $\mathrm{E}(X) = \sum_{i=1}^{\infty} \mathrm{E}(X|B_i)\,\mathrm{P}(B_i)$.

Extensions:

$$\mathrm{P}(A \cap B|C) = \mathrm{P}(A|B \cap C)\,\mathrm{P}(B|C)$$

$$\mathrm{P}(A|C) = \sum_{i=1}^{\infty} \mathrm{P}(A|B_i \cap C)\,\mathrm{P}(B_i|C),$$

$$\mathrm{E}(X|C) = \sum_{i=1}^{\infty} \mathrm{E}(X|B_i \cap C)\,\mathrm{P}(B_i|C).$$

1. Discrete-time Markov chains

It is customary to arrange the P_{ij} in a transition matrix (TM)

$$\mathbf{P} = (P_{ij}) = \begin{pmatrix} P_{00} & P_{01} & \cdots & P_{0n} \\ P_{10} & P_{11} & \cdots & P_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ P_{n0} & P_{01} & \cdots & P_{nn} \end{pmatrix}$$

Chapman-Kolmogorov equations

provide a way to compute these n-step transition probabilities.

1. $P_{ij}^{n+m} = \sum_{k \in \Omega} P_{ik}^{n} P_{kj}^{m}, \quad \forall n, m \geq 0$, all i, j
2. $\mathbf{P}^{n+m} = \mathbf{P}^{n}\mathbf{P}^{m}$

First-step analysis (Lecture 11.09)

- What is the probability that a chain is trapped in an absorbing state/class given it starts in state i?
- What is the probability that a chain reaches state j before it reaches state k?

Use

$$u_i = \mathrm{P}(A|X_0 = i)$$

where A is the event that should be fulfilled.

First-step analysis (II)

- How many steps does it take on average to reach an (absorbing) state or class the first time given it starts in state i?

Let

$$T = \min\{n \geq 0 | A\}.$$

We are interested in

$$v_i = E(T | X_0 = i)$$

where A is the event that should be fulfilled.

Classification of states

- Communication $\Rightarrow$ Equivalence class
- Irreducible?
- Recurrent? Positive-recurrent, Null-recurrent?
- Transient?
- Period?
- Ergodic

What happens for a Markov chain with a finite number of states?

What happens for a random walk with an infinite number of states?

Limiting probabilities

For an irreducible ergodic Markov chain $\lim_{n\to\infty} P_{ij}^n$ exists and is independent of i. Furthermore,

$$\pi_j = \lim_{n\to\infty} P_{ij}^n, \quad j \geq 0,$$

are the unique non-negative solution of

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad j \geq 0;$$

$$\sum_{j=0}^{\infty} \pi_j = 1$$

What is the interpretation of π_j?

Mean-time spent in transient states

Let

$$\mathbf{P}_T = \begin{pmatrix} \mathrm{P}_{11} & \mathrm{P}_{12} & \cdots & \mathrm{P}_{1t} \\ \vdots & & & \vdots \\ \mathrm{P}_{t1} & \mathrm{P}_{t2} & \cdots & \mathrm{P}_{tt} \end{pmatrix}$$

denote the transition probabilities from transient to transient states only. Then

$$\mathbf{S} = (\mathbf{I} - \mathbf{P}_T)^{-1}$$

where s_{ij} denotes the expected number of time periods the MC is in state j given it starts in state i.

Further topics

- Branching processes
- Time reversibility
- How to simulate a MC? MCMC

2. Poisson processes

The counting process $N = \{N(t), t \geq 0\}$ is called Poisson process:

- $N(0) = 0$.
- N has independent and stationary increments
- The number of events in any interval of length t is Poisson distributed with mean λt.

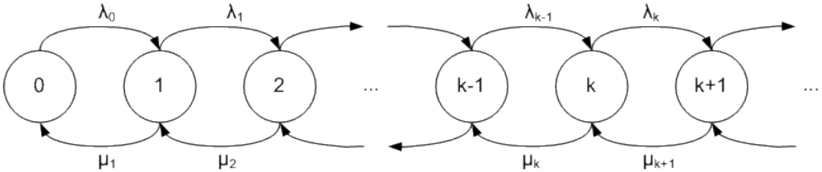
The interarrival times are independent identically exponential random variables with mean $1/\lambda$.

Note, the exponential distribution is the only distribution that is memoryless. What does this mean?

Further topics

- Waiting time distribution of n-th event
- Conditional distribution of arrival times

3. Continuous-time Markov chains



Repetition: Birth-death process

A continuous-time Markov chain $X(t)$, $t \geq$ is called birth and death process if

$$P_{i,i+1}(h) = \lambda_i h + o(h)$$

$$P_{i,i-1}(h) = \mu_i h + o(h)$$

$$P_{i,i}(h) = 1 - (\lambda_i + \mu_i)h + o(h)$$

$$P_{i,j}(h) = o(h) \quad \text{for } | j - i | \geq 2$$

with $P_{ij}(s) = P(X(t+s) = j \mid X(t) = i), i, j \in \mathcal{Z}^+$.

- $\lambda_i \geq 0$ are birth rates
- $\mu_i \geq 0$ are death rates

For $\lambda_i = \lambda$ and $\mu_i = 0$, for all i we obtain a Poisson process.

Chapman-Kolmogorov equation

For all $s, t \geq 0$:

$$P_{ij}(s+t) = \sum_{k=0}^{\infty} P_{ik}(t)\, P_{kj}(s)$$

Kolmogorov's forward equations

$$P'_{ij}(t) = \sum_{k \neq j} q_{kj}\, P_{ik}(t) - v_j\, P_{ij}(t).$$

Kolmogorov's backward equations

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik}\, P_{kj}(t) - v_i\, P_{ij}(t).$$

Limiting probabilities

If $P_j = \lim_{t\to\infty} P_{ij}(t)$ exists, P_j are given by the balance equation

$$v_j P_j = \sum_{k\neq j} q_{kj} P_k \quad \text{and} \quad \sum_j P_j = 1.$$

In particular, for a birth and death process

$$\mathrm{P}_0 = \frac{1}{\sum_{k=0}^{\infty} \theta_k} \quad \text{and} \quad \mathrm{P}_k = \theta_k P_0$$

for $k = 1, 2, \ldots$, where

$$\theta_0 = 1 \quad \text{and} \quad \theta_k = \frac{\lambda_0 \lambda_1 \cdots \lambda_{k-1}}{\mu_1 \mu_2 \cdots \mu_k}.$$

4. Queueing theory

What specifies a queue?

We looked at

- M/M/1 queues with and without finite capacity.
- M/M/k queues
- Queueing systems with bulk service
- M/G/1 queues

Fundamental quantities of interest

- L: the average number of customers in the system
- L_Q: the average number of customers waiting in the queue
- W: the average amount of time a customer spends in the system
- W_Q: the average amount of time a customer spends waiting in the queue.
- S: service time
- V: the average remaining time (or work) in the system.

Fundamental quantities of interest (II)

- P_n: limiting or long-run probability that there will be exactly n customers in the system. It is also the (long-run) proportion of time that the system contains exactly n customers.
- a_n: proportion of customers that see n persons in the system when they arrive.
- d_n: proportion of customers that see n persons in the system when they leave the system.

Little's theorem

$$L = \lambda_a W$$

$$L_Q = \lambda_a W_Q$$

$$V = \lambda_a \, \mathrm{E}(S W_Q^\star) + \lambda_a \, \mathrm{E}(S^2)/2$$

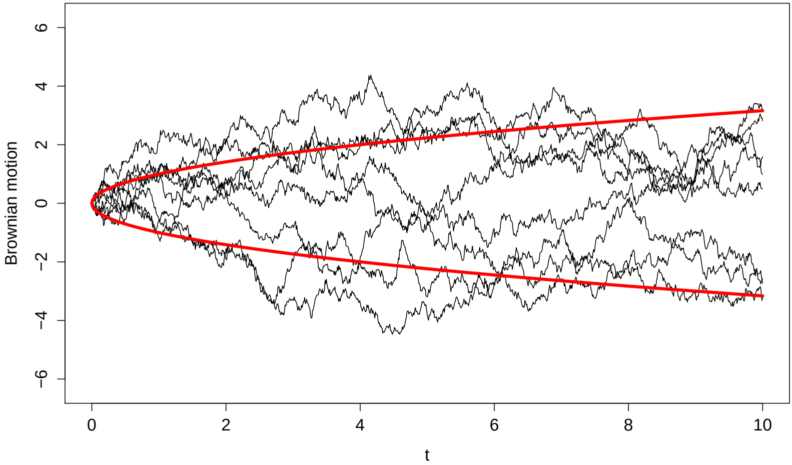
with $W_Q^\star$ denoting the time a given customer is waiting in the queue.

For Poisson arrivals we have that

$$P_n = a_n$$

This is known as PASTA-principle.

5. Brownian motion



Basics

Derived as limiting process from a symmetric discrete random walk.

A stochastic process $\{X(t), t \geq 0\}$ is said to be a Brownian motion process if

i) $X(0) = 0$

ii) $\{X(t), t \geq 0\}$ has stationary and independent increments.

iii) $X(t) \sim \mathcal{N}(0, \sigma^2 t), t > 0$.

Variations of Brownian motion

1. Let $B(t)$, $t \geq 0$ be a standard Brownian motion process. Then

   $$X(t) = \sigma B(t) + \mu t$$

   is called Brownian motion with drift.

2. Geometric Brownian motion

   $$Y(t) = \exp(\sigma B(t) + \mu t)$$
   $$\log(Y(t)) = \sigma B(t) + \mu t$$

   $Y(t)$ is log-normally distributed.

Thank you and good luck for the exam!