8.9.2 M/M/k system with infinite capacity

Assumptions as in 8.9.1

Note $X(t)$ is still a birth-death process.

$$\lambda_0 = \lambda_1 = \dots = \lambda$$

$$\mu_0 = 0, \quad \mu_1 = \mu, \dots, \mu_{k-1} = (k-1)\mu$$

$$\mu_k = \mu_{k+1} = \dots = k \cdot \mu$$

Limiting probabilities ($\Theta_0 = 1$)

$$\Theta_i = \frac{\lambda_0 \cdot \dots \cdot \lambda_{i-1}}{\mu_1 \cdot \dots \cdot \mu_i} = \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i, \quad i = 1, \dots, k-1$$

For $i = k, k+1, \dots$

$$\Theta_i = \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^i \frac{1}{k^{i-k}} = \frac{k^k}{k!}\left(\frac{\lambda}{k\mu}\right)^i$$

Thus

$$P_0 = \left(\sum_{i=0}^{\infty} \Theta_i\right)^{-1}$$

$$= \left[\sum_{i=0}^{k-1} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i + \sum_{i=k}^{\infty} \frac{k^k}{k!}\left(\frac{\lambda}{k\mu}\right)^i\right]^{-1}$$

Here, $\frac{\lambda}{k\mu} < 1$ must be fullfilled to ensure that the limiting probabilities exist.

Since

$$\sum_{i=k}^{\infty} \frac{k^k}{k!}\left(\frac{\lambda}{k\mu}\right)^i = \frac{k^k}{k!}\left(\frac{\lambda}{k\mu}\right)^k \sum_{j=0}^{\overset{(\infty-k)=\infty}{\infty}} \underbrace{\left(\frac{\lambda}{k\mu}\right)}_{<1}^{j}$$

$$= \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k \frac{1}{1-\frac{\lambda}{k\mu}} = \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k \cdot \frac{k\mu}{k\mu-\lambda}$$

$$\Rightarrow P_0 = \left[\sum_{i=0}^{k-1} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i + \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k \cdot \frac{k\mu}{k\mu-\lambda}\right]^{-1}$$

and $P_i = \Theta_i P_0$ for $i = 1, 2 \ldots$