Formulas for TMA4265 Stochastic Processes:

The law of total probability

Let $B_1, B_2, \ldots$ be pairwise disjoint events with $P(\cup_{i=1}^{\infty} B_i) = 1$. Then

$$P(A|C) = \sum_{i=1}^{\infty} P(A|B_i \cap C)P(B_i|C),$$

$$E[X|C] = \sum_{i=1}^{\infty} E[X|B_i \cap C]P(B_i|C).$$

Discrete time Markov chains

Chapman-Kolmogorov equations

$$P_{ij}^{(m+n)} = \sum_{k=0}^{\infty} P_{ik}^{(m)} P_{kj}^{(n)}.$$

For an irreducible and ergodic Markov chain, $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}$ exist and is given by the equations

$$\pi_j = \sum_i \pi_i P_{ij} \quad \text{and} \quad \sum_i \pi_i = 1.$$

For transient states i, j and k, the expected time spent in state j given start in state i, s_{ij}, is

$$s_{ij} = \delta_{ij} + \sum_k P_{ik} s_{kj}.$$

For transient states i and j, the probability of ever returning to state j given start in state i, f_{ij}, is

$$f_{ij} = (s_{ij} - \delta_{ij})/s_{jj}.$$

The Poisson process

The waiting time to the n-th event (the n-th arrival time), S_n, has the probability density

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t} \quad \text{for} \quad t \geq 0.$$

Given that the number of events $N(t) = n$, the arrival times $S_1, S_2, \ldots, S_n$ have the joint probability density

$$f_{S_1,S_2,\ldots,S_n|N(t)}(s_1, s_2, \ldots, s_n|n) = \frac{n!}{t^n} \quad \text{for} \quad 0 < s_1 < s_2 < \ldots < s_n \leq t.$$

Markov processes in continuous time

A (homogeneous) Markov process $X(t)$, $0 \leq t \leq \infty$, with state space $\Omega \subseteq \mathbf{Z}^+ = \{0, 1, 2, \ldots\}$, is called a birth and death process if

$$P_{i,i+1}(h) = \lambda_i h + o(h)$$
$$P_{i,i-1}(h) = \mu_i h + o(h)$$
$$P_{i,i}(h) = 1 - (\lambda_i + \mu_i)h + o(h)$$
$$P_{ij}(h) = o(h) \quad \text{for } |j - i| \geq 2$$

where $P_{ij}(s) = P(X(t+s) = j | X(t) = i)$, $i, j \in \mathbf{Z}^+$, $\lambda_i \geq 0$ are birth rates, $\mu_i \geq 0$ are death rates.

The Chapman-Kolmogorov equations

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s).$$

Limit relations

$$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = v_i \,, \quad \lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij} \,, \;\; i \neq j$$

Kolmogorov's forward equations

$$P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t).$$

Kolmogorov's backward equations

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

If $P_j = \lim_{t \to \infty} P_{ij}(t)$ exist, P_j are given by

$$v_j P_j = \sum_{k \neq j} q_{kj} P_k \quad \text{and} \quad \sum_j P_j = 1.$$

In particular, for birth and death processes

$$P_0 = \frac{1}{\sum_{k=0}^{\infty} \theta_k} \quad \text{and} \quad P_k = \theta_k P_0 \quad \text{for} \quad k = 1, 2, \ldots$$

where

$$\theta_0 = 1 \quad \text{and} \quad \theta_k = \frac{\lambda_0 \lambda_1 \cdot \ldots \cdot \lambda_{k-1}}{\mu_1 \mu_2 \cdot \ldots \cdot \mu_k} \quad \text{for} \quad k = 1, 2, \ldots$$

Queueing theory

For the average number of customers in the system L, in the queue L_Q; the average amount of time a customer spends in the system W, in the queue W_Q; the service time S; the average remaining time (or work) in the system V, and the arrival rate λ_a, the following relations obtain

$$L = \lambda_a W.$$

$$L_Q = \lambda_a W_Q.$$

$$V = \lambda_a E[SW_Q^*] + \lambda_a E[S^2]/2.$$

Some mathematical series

$$\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a} \quad , \qquad \sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2} \quad .$$