Solutions, review exercises 2.

ex 1

let $(e_1, \dots, e_n)$ be a basis of F. assume $\|e_i\| = 1$.

Define $\varphi_i : F \longrightarrow \mathbb{C}$ as follows:

$$\varphi_i \left(\sum_{j=1}^{n} x_j e_j \right) = x_i$$

Then φ_i is a linear form, and it is an easy check that $\|\varphi_i\| = 1$

By Hahn–Banach let $\tilde{\varphi}_i$ be the extension, for each i, of φ_i to E, such that $\|\tilde{\varphi}_i\| = \|\varphi_i\| = 1$.

Let $G = \bigcap_{i=1}^{n} N(\tilde{\varphi}_i)$. It is closed (finite intersection of closed subspaces)

Show $E = F \oplus G$.

Given $x \in E$, one writes

$$x = \underbrace{\sum_{j=1}^{n} \tilde{\varphi}_j(x)\, e_j}_{\in F} + \left(x - \sum_{j=1}^{n} \tilde{\varphi}_j(x)\, e_j \right)$$

We want to show $x - \sum_{j=1}^{n} \tilde{\varphi}_j(x) e_j \in G$.

~~Let~~ For all i,

$$\tilde{\varphi}_i \left(x - \sum_{j=1}^{n} \tilde{\varphi}_j(x) e_j \right)$$

$$= \tilde{\varphi}_i(x) - \varphi_i \left(\sum_{j=1}^{n} \tilde{\varphi}_j(x)\, e_j \right)$$

$$= \tilde{\varphi}_i(x) - \tilde{\varphi}_i(x) \quad \text{(by definition of } \varphi_i\text{)}$$

$$= 0$$

so $\forall i, \quad \left(x - \sum_j \tilde{\varphi}_j(x)\, e_j\right) \in \mathcal{N}(\varphi_i)$

so $\quad \left(x - \sum_j \tilde{\varphi}_j(x)\, e_j\right) \in G.$

. ~~The~~ So $\quad E = F + G$

. The sum is direct:

if $x \in F \cap G$,

$x = \sum_{i=1}^{n} x_i e_i \quad$ because $x \in F$

$\forall i, \; x_i = 0 \quad$ because $\varphi_i(x) = 0$ (because $x \in G$

so $x = 0$

So $\quad E = F \oplus G.$

<u>ex2</u> First, if $f : F \to \mathbb{C}$ is bounded, there is a unique extension to $\overline{F}$ by continuity, so we may assume that F is closed.

<u>proof</u>: if $x \in \overline{F}$, let $x_n \to x$.
Then (x_n) is Cauchy, so $(f(x_n))$ is Cauchy and define its limit to be $f(x)$.
It is unique (and well defined)

Now, if F is closed, $H = F \oplus F^{\perp}$

define $\tilde{f}(x_F + x_{F^\perp}) = f(x_F) + 0.$

- it is norm-preserving: easy

- it is unique

Assume that g is another norm-preserving extension of f, such that $\|f\| = \|g\|$

Then $\exists\, y \in F^\perp$ such that $g(y) \neq 0$.

Let us derive a contradiction from that.

We may assume $\|y\| = 1$.

Let (x_n) be a sequence such that

$$\cdot\ \|x_n\| = 1 \qquad \forall n$$

$$\cdot\ |f(x_n)| \longrightarrow \|f\|$$

Up to multiplying x_n by $\overline{\dfrac{f(x_n)}{|f(x_n)|}}$, we may assume $f(x_n) \longrightarrow \|f\|$.

For any θ, define

$$x_{\theta,n} = (\cos\theta) x_n + \sin\theta\, y$$

Then $\|x_{\theta,n}\| = 1$ because $x_n \perp y$

and $\|x_n\| = 1$, $\|y\| = 1$

and $\cos^2\theta + \sin^2\theta = 1$.

and $g(x_{\theta,n}) = \cos\theta\, g(x_n) + \sin\theta\, g(y)$, but $f(x_n) = g(x_n)$

$$\longrightarrow \cos\theta \|f\| + \sin\theta\, g(y)$$

for small θ, this quantity is

$$\|f\| + \theta\, g(y) + o(\theta) > \|f\|$$

for a good value of θ (because $g(y) \neq 0$)

so $\|g\| > \|f\|$.

ex 3

1) $\|Tx\|^2 = \langle Tx, Tx\rangle$
$= \langle T^*Tx, x\rangle$
$= \langle TT^*x, x\rangle$
$= \langle T^*x, T^*x\rangle = \|T^*x\|^2$

2) so $\|Tx\|=0 \iff \|T^*x\|=0$
so $N(T) = N(T^*)$

3) assume T is injective.
Then T^* is injective
Assume $R(T)$ is not dense
So there is $y \in R(T)^\perp \setminus \{0\}$
so $\forall x \in H$, $\langle Tx, y\rangle = 0$
so $\langle x, T^*y\rangle = 0$ $(\forall x \in H)$
so $y \in N(T^*) = N(T)$
so $y = 0$, contradiction
~~so y ∈~~ $R(T)^\perp = \{0\}$
so $(R(T)^\perp)^\perp = \overline{R(T)} = H$.

Conversely, assume $R(T)$ is dense, so $R(T)^\perp = \{0\}$
Let x such that $Tx = 0$. so $T^*x = 0$
So $\forall y$, $\langle T^*x, y\rangle = 0$

So $\forall y, \quad \langle x, Ty \rangle = 0$

so $x \in R(T)^{\perp}$ so $x = 0$

so T is injective.

<u>ex4</u> $r_\sigma(T) = \lim_{n \to \infty} \|T^n\|^{1/n}$

If T is nilpotent, this sequence is stationnary, equal to 0 after a point.

So $r_\sigma(T) = 0$, so $\sigma(T) = \{0\}$

<u>ex5</u>

$L^2([0,1])$ is a Hilbert space with $\langle f, g\rangle = \int_0^1 f\bar{g}\, d$

$\forall x$, let χ_x be the function

$$\chi_x(y) = \begin{cases} 1 & \text{if } y \leq x \\ 0 & \text{if } y > x \end{cases}$$

$\chi_x \in L^2$, and

$$|Vf(x)| = |\langle f, \chi_x \rangle|$$

$$\leq \|f\|_{L^2} \|\chi_x\|_{L^2} \text{ by Cauchy-Schwartz}$$

but $\|\chi_x\|_{L^2}^2 = \int_0^1 \chi_x^2(y)\,dy$

$$= \int_0^x dy = x \leq 1$$

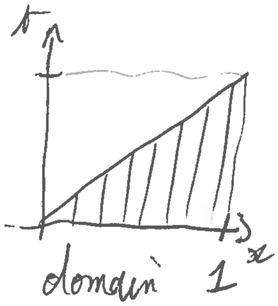
so $|Vf(x)| \leq \|f\|_{L^2}$ ~~and~~ for all x.

so $\int_0^1 |Vf(x)|^2 dx \leq \|f\|_{L^2}^2$

so $\|Vf\|_{L^2} \leq \|f\|_{L^2}$ and V is bounded.

2) $\forall f, g$, $\langle Vf, g\rangle = \int_0^1 Vf(x)\,\overline{g(x)}\,dx$

$$= \int_0^1 \int_0^x f(t)\,dt\,\overline{g(x)}\,dx$$



$$= \int_0^1 \int_0^x f(t)\,\overline{g(x)}\,dt\,dx$$

Fubini

$$= \int_0^1 \int_t^1 f(t)\,\overline{g(x)}\,dx\,dt$$

$$= \int_0^1 f(t)\left(\overline{\int_t^1 g(x)\,dx}\right)dt$$

so $V^* g(t) = \int_t^1 g(x)\,dx$.

ex 6 $T \mapsto T^{-1}$ is continuous on its open domain in $B(H)$.
(So its complement is closed).

so if $T_n - \alpha_n I$ is not invertible for all n,
assume that $T - \alpha I$ is invertible.
Then if S is close to $T - \alpha I$, S is also invertible

But since $T_n - \alpha_n I \to T - \alpha I$,
it means $T_n - \alpha_n I$ is invertible for n big enough

Contradiction

So $T - \alpha I$ is not invertible

so $\alpha \in \sigma(T)$.

ex 7 Remark that

$$A^{-1}TA - \lambda I = A^{-1}TA - \lambda A^{-1}A$$
$$= A^{-1}(T - \lambda I) A.$$

and $A^{-1}SA$ is invertible
iff S is invertible (given that A is invertible).

ex 8 1) For all $x \in H$, and all $N \in \mathbb{N}$,

$$x = \sum_{n=1}^{N} P_n x + \left(x - \sum_{n=1}^{N} P_n x\right)$$

If P_n is the projection onto E_n,

$$\sum_{n=1}^{N} P_n x \in \bigoplus_{n=1}^{N} E_n$$ (seen in class: finite sums of projections)

and $\forall k \; (k \leq N)$, $P_k\left(x - \sum_{n=1}^{N} P_n x\right) = 0$ (direct computati so $\left(x - \sum_{n=1}^{N} P_n x\right) \in N(P_k) = E_k^{\perp} \quad \forall k \leq N$

so $\left(x - \sum_{n=1}^{N} P_n x\right) \in \left(\bigoplus_{n=1}^{N} E_n\right)^{\perp}$

so by Pythagoras' theorem,

$$\|x\|^2 = \left\|\sum_{n=1}^{N} P_n x\right\|^2 + \left\|x - \sum_{n=1}^{N} P_n x\right\|^2$$

so $\left\|\sum_{n=1}^{N} P_n x\right\|^2 \leqslant \|x\|^2$

Furthermore, since $P_i x \perp P_j x$ as soon as $i \neq j$,

$$\left\|\sum_{n=1}^{N} P_n x\right\|^2 = \sum_{n=1}^{N} \|P_n x\|^2 \leqslant \|x\|^2$$

so in particular, $\sum_{n=1}^{N} \|P_n x\|^2$ converges: $(\|P_n x\|^2)$ is summable

now if (α_n) is a bounded sequence, (say bounded by M),

$$\cancel{\left\|\sum_{n=1}^{N} \alpha_n P_n x\right\|^2 \leq \sum_{n=1}^{N} |\alpha_n| \, \|P_n x\|^2}$$

$$\cancel{\leq M \sum_{n=1}^{N} \|P_n x\|^2 \leq M \|x\|^2}$$

~~con~~

so $\left\|\sum_{n=i}^{j} \alpha_n P_n x\right\|^2 = \sum_{n=i}^{j} |\alpha_n| \, \|P_n x\|^2$

$$\leqslant M \sum_{n \geqslant i} \|P_n x\|^2 \xrightarrow[i \to \infty]{} 0$$

so $\left(\sum_{n=1}^{N} \alpha_n P_n x\right)_N$ is Cauchy, so it converges in H.

so $\left(\sum_{n\geq 1} \alpha_n P_n\right)$ makes sense as the limit of $\left(\sum_{n=1}^{N} \alpha_n P_n\right)_{N\in\mathbb{N}}$ in the strong operator topology

2) no (example: let $(e_1, \dots, e_n, \dots)$ be a basis of H, and $P_i(x) = \langle x, e_i\rangle e_i$ (projection on $\mathrm{Span}(e_i)$))

$\alpha_n = 1 \quad \forall n.$

Then $\left\| \sum_{n=1}^{N+1} \alpha_n P_n - \sum_{n=1}^{N} \alpha_n P_n \right\|$

$= \|P_{N+1}\| = 1$

so $\left(\sum_{n=1}^{N} \alpha_n P_n\right)_N$ is not Cauchy in $B(H)$ so it does not converge in norm