Solutions for review exercises 1

<u>ex 1</u>. Let $\lambda_1 \neq \lambda_2$ be the elements of $\sigma(T)$. $(\lambda_i \in \mathbb{R})$

Let $P = \dfrac{1}{\lambda_2 - \lambda_1}(T - \lambda_1 I)$

Then $T = \lambda_1 I + (\lambda_2 - \lambda_1) P$

We need to check that P is a projection.

Clearly, $P^* = P$

method 1

Using the theorem on functional calculus, there is an algebra morphism

$$\mathcal{C}(\sigma(T)) \longrightarrow B(H)$$
$$f \longmapsto f(T)$$

let $f(x) = \dfrac{1}{\lambda_2 - \lambda_1}(x - \lambda_1)$

Then $f(\lambda_1) = 0$, $f(\lambda_2) = 1$

so $f^2 = f$ <u>when restricted on $\sigma(T)$</u>

$(f(\lambda_1))^2 = 0^2 = 0 = f(\lambda_1)$, $\quad f(\lambda_2)^2 = 1^2 = 1 = f(\lambda_2)$

so $P^2 = (f(T))^2 = (f^2)(T) = f(T) = P$.

so P is a projection.

~~variant.~~ ~~let~~ $\cancel{f(x) = \dfrac{1}{\lambda_2 - \lambda_1}(T - \lambda_1 I)}$

variant let $f(x) = \frac{1}{d_2 - d_1}(x - d_1)$

method 2.

f is a polynomial. By the spectral mapping theorem $\sigma(f(T)) = f(\sigma(T))$

so $\sigma(P) = \{0, 1\}$

so $\sigma(P - P^2) = \{0\}$ and $I - P^2$ is self-adjoint.

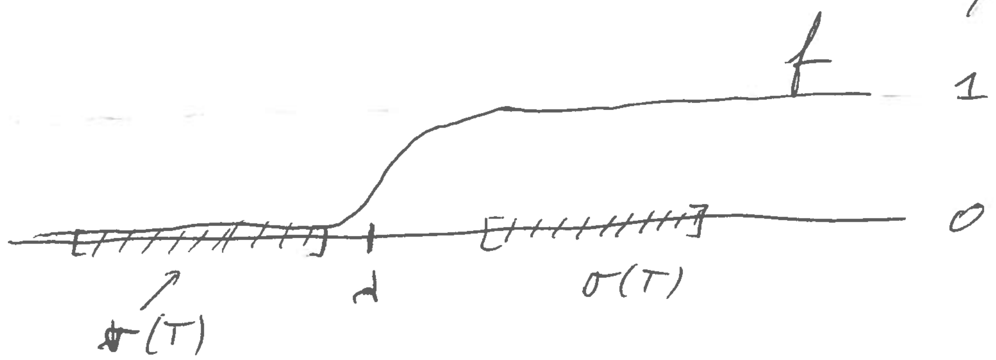
~~so $\langle (P-P^2)x, x\rangle = 0 \quad \forall x \in H$~~

and we have seen that for self-adjoint operators, $\|P - P^2\| = r_\sigma(P - P^2) = 0$

so $P = P^2$ and P is a projection.

- if $\sigma(T)$ consists of more than one interval there is $f \in \mathcal{C}(\mathbb{R})$ such that $\forall \varepsilon > 0$
  - $f|_{]-\infty, d-\varepsilon]} = 0$
  - $f|_{[d+\varepsilon, +\infty[} = 1$
  - f is continuous

In particular, let's choose such an f for $\varepsilon = d(d, \sigma(T)) > 0$ (because $d \notin \sigma(T)$ and $\sigma(T)$ is compact)



Then, $\forall \mu \in \sigma(T)$, $f(\mu) = 0$ or $f(\mu) = 1$

so $f^2 = f$ <u>when restricted on $\sigma(T)$</u>

so $(f(T))^2 = f^2(T)^{\ast} = f(T)$

and $f(T)^* = \bar{f}(T^*) = f(T)$

because T is self-adjoint and f is real-valued

so $f(T) =: P$ is a projection.

$\sigma(f(T)) = f(\sigma(T)) = \{0, 1\}$

so $f(T) \neq I$

$f(T) \neq 0$.

<u>ex 2</u> Clearly, if S is a symmetry, $S^2 = I$

Plus, $\forall x = x_E + x_{E^\perp}$ $\forall y = y_E + y_{E^\perp}$,

we have $\langle Sx, y \rangle$

$= \langle x_E - x_{E^\perp}, y_E + y_{E^\perp} \rangle$

$= \langle x_E, y_E \rangle - \langle x_{E^\perp}, y_{E^\perp} \rangle$

$= \langle x_E + x_{E^\perp}, y_E - y_{E^\perp} \rangle = \langle x, Sy \rangle$

so $S^* = S$.

Conversely, if ~~S H~~ $S^2 = I$ and $S = S^*$, we want to define :
- a space E on which S does nothing (acts like I)
- a space F on which S acts like $-I$
- show that $F = E^\perp$.

so let $E = \mathcal{N}(S-I)$, $F = \mathcal{N}(S+I)$

A computation shows $(S-I)(S+I) = (S+I)(S-I)$
$= S^2 - I^2 = 0$

so $\mathcal{R}(S+I) \subseteq \mathcal{N}(S-I) = E$
and $\mathcal{R}(S-I) \subseteq \mathcal{N}(S+I) = F$

Furthermore, any $x \in H$ can be written

$$x = \frac{x+Sx}{2} + \frac{x-Sx}{2} = \underbrace{\tfrac{1}{2}(I+S)x}_{\in \mathcal{R}(S+I) \subseteq E} + \underbrace{\tfrac{1}{2}(I-S)x}_{\in F}$$

so $H = E + F$ (and E, F are closed because they are nullspaces of ~~self~~ bounded operators)

now if $x \in E \cap F$,

$$x = \underbrace{\tfrac{1}{2}(I+S)x}_{\substack{=0 \text{ because} \\ x \in F = \mathcal{N}(I+S)}} + \underbrace{\tfrac{1}{2}(I-S)x}_{=0 \text{ because } x \in E = \mathcal{N}(I-S)}$$

so $H = E \oplus F$.

(note: it also proves that $\mathcal{N}(S-I) = \mathcal{R}(S+I)$
and $\mathcal{N}(S+I) = \mathcal{R}(S-I)$)

Finally, let $x \in E$, $y \in F$

Then $x \in N(S-I) = R(S+I)$

$$\text{so } x = \frac{1}{2}(I+S)x \quad \Big(\underbrace{+\frac{1}{2}(I-S)x}_{=0}\Big)$$

$$\text{so } \langle x, y\rangle = \left\langle \frac{1}{2}(I+S)x, y\right\rangle$$

$$= \left\langle x, \frac{1}{2}(I+S)y\right\rangle \qquad (I+S)^* = I+S$$

$$= 0 \text{ because } y \in N(I+S)$$

so $E \perp F$.

Since they are closed and in direct sum, $F = E^\perp$.

and if $x \in H$, $x = x_E + x_{E^\perp}$

$$S x_E = x_E \qquad (\text{because } (I-S)x_E = 0)$$

$$S x_{E^\perp} = -x_{E^\perp} \quad (\text{because } (I+S)x_{E^\perp} = 0)$$

$$\text{so } \quad Sx = x_E - x_{E^\perp}.$$

<u>ex 3</u>

1) if $x_n \xrightarrow{w} x$, then $(\|x_n\|)_{n\in\mathbb{N}}$ is bounded. (say by M)

Let us evaluate

$$|\langle x_n, y_n\rangle - \langle x, y\rangle| = |\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle|$$

$$\leq |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle|$$

$$\leq \underbrace{\|x_n\|}_{\leq M} \underbrace{\|y_n - y\|}_{\to 0} + \underbrace{|\langle x_n - x, y\rangle|}_{\to 0 \text{ by definition of weak convergence}}$$

2) let $x_n = y_n = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$

n^{th} position

in $H = \ell^2(\mathbb{N})$

Then $x_n \xrightarrow{w} 0$ but $\langle x_n, y_n \rangle = 1 \ \forall n$.

$y_n \xrightarrow{w} 0$

<u>ex 4</u> very similar to theorem 4.9.6 in the text.

<u>ex 5</u>

1) $(\Rightarrow)$ assume $T_n \xrightarrow{\|\cdot\|} 0$

Then $|\langle T_n x_n, y_n \rangle| \leq \|T_n x_n\| \|y_n\|$ (Cauchy-Schwarz)

$$\leq \underbrace{\|T_n\|}_{\to 0} \underbrace{\|x_n\|}_{\text{bounded}} \underbrace{\|y_n\|}_{\text{bounded}}$$

$$\longrightarrow 0$$

$(\Leftarrow)$ For any given sequence (x_n) bounded,

pick $y_n = \begin{cases} Tx_n / \|Tx_n\| & \text{if } Tx_n \neq 0 \\ 0 & \text{else} \end{cases}$

Then y_n is bounded, and $\langle T_n x_n, y_n \rangle = \|Tx_n\|$

so for any bounded sequence (x_n), $\|T_n x_n\| \to 0$.

In particular, for all n, pick x_n such that

$\|T_n x_n\| \geq \|T_n\| / 2$ (and $\|x_n\| = 1$)

Then $\|T_n\| \leq 2 \|T_n x_n\| \longrightarrow 0$

so $\|T_n\| \to 0$

2) ($\Rightarrow$) assume $T_n \xrightarrow{SOT} 0$

Then $\forall x \in H$, $T_n x \to 0$

so forall bounded sequence (y_n), and all sequence $x_n \to x$,

$$\begin{aligned}
&|\langle T_n x_n, y_n\rangle| \\
&= |\langle T_n x, y_n\rangle + \langle T_n(x_n - x), y_n\rangle| \\
&\leq |\langle T_n x, y_n\rangle| + |\langle T_n(x_n - x), y_n\rangle| \\
&\leq \underbrace{\|T_n x\|}_{\to 0} \cdot \underbrace{\|y_n\|}_{\text{bounded}} + \|T_n\| \underbrace{\|x_n - x\|}_{\to 0} \underbrace{\|y_n\|}_{\text{bounded}}
\end{aligned}$$

now $\forall x$, $T_n x \to 0$ so by uniform boundness theorem, $(\|T_n\|)$ is bounded

so $|\langle T_n x_n, y_n\rangle| \to 0$.

($\Leftarrow$) Conversely, $\forall x$, let (x_n) be the sequence constantly equal to x,

and $y_n = \begin{cases} \dfrac{T_n x}{\|Tx\|} & \text{if } T_n x \neq 0 \\ 0 & \text{if } T_n x = 0 \end{cases}$

Then $x_n \to x$, (y_n) is bounded,

so $\langle T_n x_n, y_n\rangle \to 0$

but $\langle T_n x_n, y_n\rangle = K \|T_n x\| \to 0$

so $T_n \xrightarrow{SOT} 0$.

3) $(\Leftarrow)$ if $T_n \xrightarrow{WOT} 0$ it means

$$\forall x, y \in H, \ \langle T_n x, y\rangle \longrightarrow 0$$

so if $x_n \to x$, $y_n \to y$,

$$|\langle T_n x_n, y_n\rangle| = |\langle T_n x_n, y_n\rangle - \langle T_n x, y_n\rangle + \langle T_n x, y_n\rangle - \langle T_n x, y\rangle + \langle T_n x, y\rangle|$$

$$\leq |\langle T_n(x_n - x), y_n\rangle| + |\langle T_n x, y_n - y\rangle| + |\langle T_n x, y\rangle|$$

$$\leq \|T_n\| \, \|x_n - x\| \, \|y_n\| + \|T_n\| \, \|x\| \, \|y_n - y\| + |\langle T_n x, y\rangle|$$

now: $\|x_n - x\| \to 0$, $\|y_n - y\| \to 0$,
$\langle T_n x, y\rangle \to 0$

so if we can show that $(\|T_n\|)_n$ is bounded, all the expression tends to 0.

- $\forall x, y$ $\langle T_n x, y\rangle \to 0$, so $\langle T_n x, y\rangle$ is bounded.

so $(\langle y, T_n x\rangle)_n$ is bounded $\forall x, y$

By uniform boundedness, the sequence of linear forms $\varphi_n : (y \mapsto \langle y, T_n x\rangle)$ is bounded. But $\|\varphi_n\| = \|T_n x\|$ so for all x, $(\|T_n x\|)_n$ is bounded. So by uniform boundedness again, $(\|T_n\|)_n$ is bounded.