TMA 4230 - Review exercises I (April 25th)

1 Spectral projections

Exercises 9.8.3, 9.8.8, 9.9.8, 9.9.9 of the textbook.

2 Functional calculus for self-adjoint operators

Exercise 1

Let $T : H \longrightarrow H$ be a bounded self-adjoint operator.

- Show that if the spectrum of T consists of two points, then there is a non-zero projection $P \in \mathcal{B}(H)$ such that

  $$T = aP + bI,$$

  for some real numbers a and b ($a \neq 0$).

- Assume that the spectrum of T consists of more than one interval: there exists $\lambda \in [m, M]$ (in the notations of the textbook), such that $\lambda \notin \sigma(T)$. Show that there is a continuous function f such that $f(T)$ is a non-trivial projection (not 0 or I).

3 Review exercises

The following exercise uses methods which we used in class for projections.

Exercise 2

A bounded operator S on a Hilbert space H is a symmetry if there is a decomposition $H = E \oplus E^{\perp}$ such that in this decomposition,

$$S(x_E + x_{E^{\perp}}) = x_E - x_{E^{\perp}}.$$

Show that S is a symmetry if and only if $S^2 = I$ and $S = S^*$.

For these three exercises, you may want to review the definition of weak convergence, as well as consequences of the Banach–Steinhaus (or uniform boundedness principle) for weak convergence. Namely: a weakly converging sequence is bounded.

Exercise 3

Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ two sequences in a Hilbert space H.

- Assume x_n converges to x weakly, and y_n converges to y for the norm. Show that $\langle x_n, y_n\rangle$ converges to $\langle x, y\rangle$ in $\mathbb{C}$. [Hint: use in particular that a weakly converging sequence is bounded].

- Show that the previous point does not hold if both x_n and y_n converge weakly (find a counter example).

Exercise 4

Let T_n be a sequence of bounded linear operators from a Banach space E to a normed vector space F, let $T \in \mathcal{B}(E, F)$ be another bounded linear operator, and let D be a dense subspace of E. Then the following are equivalent:

- T_n converges in the strong operator topology of $\mathcal{B}(E, F)$ to T.

- T_n is bounded in the operator norm (i.e. $\|T_n\|_{op}$ is bounded), and the restriction of T_n to D converges in the strong operator topology to the restriction of T to D.

Exercise 5

Let H be a Hilbert space, and (T_n) be a sequence of bounded linear operators on H.

- Show that $T_n \to 0$ in the operator norm topology if and only if $\langle T_n x_n, y_n\rangle \to 0$ for any bounded sequences $x_n, y_n \in H$.

- Show that $T_n \to 0$ in the strong operator topology if and only if $\langle T_n x_n, y_n\rangle \to 0$ for any convergent sequence $x_n \in H$ and any bounded sequence $y_n \in H$.

- Show that $T_n \to 0$ in the weak operator topology if and only if $\langle T_n x_n, y_n\rangle \to 0$ for any convergent sequences $x_n, y_n \in H$.