# TMA 4230 - Review exercises II (April 29th)

## 1 On Hahn–Banach

**Exercise 1**

Using the Hahn–Banach theorem, show that for any finite-dimensional subspace $F$ in a normed space $E$, there is a closed subspace $G$ of $E$ such that

$$E = F \oplus G.$$

**Exercise 2**

Let $F$ be a subspace of a Hilbert space $H$. Let $f$ be a bounded functional on $F$. Give explicitely one norm-preserving extension of $f$ to $H$. Prove that it is unique.

## 2 Operators on a Hilbert space

**Exercise 3**

Let $T$ be a normal bounded operator on a complex Hilbert space $H$ (that is $TT^* = T^*T$).

1. Show that for all $x \in H$, $\|Tx\| = \|T^*x\|$.
2. What can you say about the nullspaces of $T$ and $T^*$?
3. Show that $T$ is injective if and only if its image is dense.

**Exercise 4**

An operator $T$ is *nilpotent* if $T^n = 0$ for some $n$. Show that the spectrum of a nilpotent operator is $\{0\}$. Is the converse true?

**Exercise 5**

Consider the Volterra operator defined on $L^2([0,1])$:

$$Vf(x) = \int_0^x f(t)dt.$$

1. Show that it is a bounded operator;
2. What is the adjoint $V^*$ of $V$?

**Exercise 6**

Let $T_n$ be a sequence of linear bounded operators on a complex Hilbert space, such that $T_n \to T$ as $n \to +\infty$. For all $n$, let $\alpha_n \in \sigma(T_n)$, such that $\alpha_n \to \alpha$. Is it the case that $\alpha \in \sigma(T)$?

**Exercise 7**

Let $T$ be an operator in $\mathcal{B}(H)$. Show that for any invertible operator $A \in B(H)$,

$$\sigma(A^{-1}TA) = \sigma(T).$$

**Exercise 8**

Let $(P_n)$ be a sequence of pairwise orthogonal projections, and $(\alpha_n)$ be a bounded sequence of complex numbers. Let $T = \sum \alpha_n P_n$, that is for all $x \in H$,

$$Th = \sum_{n \in \mathbb{N}} \alpha_n P_n h.$$

1. Why does the sum make sense?
2. Do the partial sums converge in norm to $T$?