TMA 4230 - Exercises for March 4th, 2013

Some properties that I didn't specifically mention in class:

- A closed subset of a compact space is compact;
- The image of a compact subspace by a continuous map is compact;
- A function is continuous iff the inverse image of any closed set is closed;
- A continuous, bijective function defined on a compact set is a homeomorphism (its inverse is continuous).

Exercise 1

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two Hausdorff topologies on X, such that $\mathcal{T}_2$ is finer than $\mathcal{T}_1$ (that is: $\mathcal{T}_1 \subset \mathcal{T}_2$).

- Show that if F is $\mathcal{T}_1$ closed, it is $\mathcal{T}_2$ closed.
- Show that if, for some topological space Y, $f : Y \to (X, \mathcal{T}_2)$ is continuous, then $f : Y \to (X, \mathcal{T}_1)$ is continuous.
- Show that if K is a $\mathcal{T}_2$-compact subset of X, then it is $\mathcal{T}_1$-compact.
- If X is T_2-compact, show that $\mathcal{T}_1 = \mathcal{T}_2$.

Exercise 2

Let X and Y be topological spaces, and consider the product space $X \times Y$, with the product topology. Show that if $A \subset X$ and $B \subset Y$, then $\overline{A \times B} = \overline{A} \times \overline{B}$.

Exercise 3

Let X and Y be topological spaces, and f a map from X to Y. Assume Y is Hausdorff and f is continuous, and show that the graph

$$\mathcal{G}(f) = \{(x, y) \in X \times Y \ ; \ f(x) = y\}$$

is closed in $X \times Y$.

Assume X and Y are both compact and Hausdorff, and $\mathcal{G}(f)$ is closed in $X \times Y$. Then f is continuous.
Hint: if you want to prove it using nets, you can use this property which we didn't prove in class: if (x_i) is a net in a compact space such that (x_i) has a unique cluster point x, then (x_i) converges to x.

Exercise 4 (*)

Let E be a separable normed space. Show that the unit ball of the dual E' is sequentially compact for the weak-$*$-topology: given $(f_n)_{n\in\mathbb{N}}$ a sequence of linear operators of norm less than one, there is a sub-sequence which converges to $f \in E'$ for the weak-$*$ topology.

Hint: you can consider a dense sequence $(x_k)_{k\in\mathbb{N}}$ in the unit ball of E, and build a sub-sequence of the f_n's such that $(f_n(x_k))_n$ converges for all k.

Exercise 5 (*)

Let E be a Banach space.

1. Show that for all $f_0 \in E'$, the function defined on E' by $f \mapsto \|f - f_0\|$ is lower semicontinuous (that is the inverse image of $[0, \alpha]$ is closed for all $\alpha \geq 0$), for the weak-$*$ topology.

2. Show that a lower semicontinuous function defined on a compact set attains its minimum.

3. Let F be a closed subspace of E', and $f_0 \notin F$. Then, show that there exists $g_0 \in F$ such that $\|f_0 - g_0\| = d(f_0, F)$.

Erratum: The space E should be assumed to be reflexive. Then it is possible to prove the following fact:

- A norm-closed subspace is weak-closed, and therefore weak-$*$-closed.

Actually, assuming reflexivity is necessary: a Banach space is reflexive if and only if every closed hyperplane F is proximal (in the sense that for any $x \in E$, there is $x_F \in F$ such that $\|x - x_F\| = d(x, F)$). (R. C. James, Weak compactness and reflexivity, Israel J. of Math., 1964).