Problem 1 Let

$$X = \left\{ x = (x(n))_{n=1}^{\infty} \ : \ \sum_{n=1}^{\infty} n|x(n)|^2 < \infty \right\}.$$

Show that X is a dense subspace of $\ell^2(\mathbb{N})$. Is X closed?

Problem 2 Consider the functional φ acting on sequences $x = (x(n))_{n=1}^{\infty}$, given by

$$\varphi(x) = \sum_{n=1}^{\infty} \frac{x(n)}{\sqrt{n}}.$$

For which p, $1 \leq p \leq \infty$, is $\varphi \colon \ell^p(\mathbb{N}) \to \mathbb{C}$ continuous? When continuous, what is the norm of φ?

Problem 3 Let X be a Banach space, and let $x^* \in X^*$. Assume that $(x_n^*)_{n=1}^{\infty} \subset X^*$ is a sequence such that $\sup_n \|x_n^*\| < \infty$ and suppose that there exists a dense subspace $D \subset X$ such that $x_n^*(x) \to x^*(x)$ for every $x \in D$. Prove that $(x_n^*)_{n=1}^{\infty}$ converges weak-star to x^*.

Problem 4 Let X be a reflexive Banach space, and let Y be a subspace of X. Show that $(Y^{\perp})^{\perp}$ coincides with the closure of Y in X.

Problem 5 Let X be a reflexive Banach space. A sequence $(T_n)_{n=1}^{\infty}$ of bounded operators $T_n \colon X \to X$ is said to converge in the weak operator topology if

$$\lim_{n \to \infty} x^*(T_n x)$$

exists for every $x \in X$ and $x^* \in X^*$. Show that there exists a bounded operator $T \colon X \to X$ such that $T_n x \to Tx$ weakly for every $x \in X$.

Problem 6 Let X be a topological vector space.

a) Given an open neighbourhood U of the origin, show that there is an open neighbourhood V of the origin such that $V + V \subset U$.

b) Suppose that $F \subset X$ is a closed subset of X, and that $x \notin F$. Show that there are disjoint open sets O_1 and O_2 such that $x \in O_1$ and $F \subset O_2$.

Problem 7

a) State the open mapping theorem and the closed graph theorem.

b) Prove that the open mapping theorem implies the closed graph theorem.

Problem 8 Let $(f_n)_{n=1}^{\infty}$ be a sequence in $C([0,1])$.

a) Prove that $f_n \to 0$ weakly if and only if the sequence $(f_n)_{n=1}^{\infty}$ is bounded and $f_n(s) \to 0$ for every $s \in [0,1]$.

b) Give an example of a sequence $(f_n)_{n=1}^{\infty}$ which tends weakly to 0, but which is not norm-convergent.

Problem 9 Suppose that $K \in C([0,1]^2)$, and consider the corresponding integral operator,

$$Tf(x) = \int_0^1 K(x,y) f(y)\, dy.$$

Prove that $T\colon C([0,1]) \to C([0,1])$ is compact.

Hint: By the Stone–Weierstrass theorem, $K(x,y)$ can be approximated uniformly on $[0,1]^2$ by a sequence $(p_n)_{n=1}^{\infty}$ of polynomials $p_n(x,y)$.

Problem 10 Define a kernel on $[0,1]^2$ by

$$K(x,y) = \begin{cases} (x-1)y \text{ if } y \le x, \\ x(y-1) \text{ if } y > x, \end{cases}$$

and consider the corresponding integral operator,

$$Tf(x) = \int_0^1 K(x,y) f(y)\, dy.$$

Show that $T\colon L^2([0,1]) \to L^2([0,1])$ is compact and symmetric, and compute its spectrum $\sigma(T)$.