Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4230 Functional Analysis
Spring 2023

Practice Exam Questions

1 Let X and Y be Banach spaces. Suppose that $T : X \to Y$ is a bounded linear operator whose range $\operatorname{ran} T$ is dense in Y. Prove that $T^* \colon Y^* \to X^*$ is injective.

2 Let X be a Banach space and let $T : X \to X$ be a bounded linear operator. Suppose that $T^n = 0$ for some positive integer n. Prove that $I - T$ is invertible and that

$$(I-T)^{-1} = \sum_{k=0}^{n-1} T^k.$$

3 Let X be a topological vector space, and let Y be a closed subspace of X. Prove that the quotient space X/Y, equipped with the quotient topology, is a topological vector space.

4 Let X be a Banach space. Prove that $\mathcal{L}(X)$, the space of all bounded linear operators on X, is also a Banach space.

5 Let X be a Banach space, and let $T : X \to X$ be a bounded linear operator. Prove that if $T^2 = T$ and $T \neq 0$, then $\|T\| \geq 1$.

6 Let X be a reflexive Banach space, and let T be a bounded linear operator on X. Prove that T is compact if and only if it maps every weakly convergent sequence $(x_n)_{n=1}^{\infty}$ into a norm-convergent sequence $(Tx_n)_{n=1}^{\infty}$.

7 In a Hilbert space H, suppose that $x_n \to x$ in the weak topology, and that $\|x_n\| \to \|x\|$. Prove that $x_n \to x$ in the norm topology of H.

8 Let $K \in C([0,1] \times [0,1])$, and let T be the corresponding integral operator,

$$Tf(x) = \int_0^1 K(x,y) f(y)\, dy.$$

Prove that $T \colon L^1([0,1]) \to C([0,1])$ is bounded with norm

$$\|T\| = \sup_{x,y\in[0,1]} |K(x,y)|.$$

9 Let $T\colon L^2([0,1]) \to L^2([0,1])$ be the integral operator

$$Tf(x) = \int_0^1 \min\{x, y\} f(y)\, dy.$$

Show that T is compact and self-adjoint, and find the spectrum $\sigma(T)$ of T.