A convex explanation of $(L^p)^* = L^q$ for $p > 1$

Xu Wang

Fix $1 < p < \infty$. Define $q > 1$ such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

This note is an attempt to explain why we can expect an L^q representation for a bounded linear functional on L^p, where by L^p we mean the completion of $C[0,1]$ under the L^p norm

$$||u||_{L^p} := \left(\int_0^1 |u(x)|^p \, dx \right)^{1/p}.$$

Let us start by looking at the $p = q = 2$ case. Let f be a bounded ($\mathbb{R}$-)linear functional on L^2. Our aim is to find $u_f \in L^2$ such that

$$(\star) \qquad f(u) = (u, u_f).$$

The main idea is to look at

$$\ker f := \{u \in L^2 : f(u) = 0\}.$$

If $\ker f = L^2$ then $f \equiv 0$ and one can simply take $u_f = 0$. The non-trivial case is when $\ker f$ is a proper subspace of L^2, so we can take $u_0 \in L^2$ such that $f(u_0) \neq 0$.

Fact 1. $L^2 = \ker f + \mathbb{R} u_0$. *In fact, for every* $u \in L^2$, *we have*

$$f(u) = \frac{f(u)}{f(u_0)} f(u_0),$$

hence $u - \frac{f(u)}{f(u_0)} u_0 \in \ker f$.

Fact 2. *There exists a unique* $\hat{u} \in \ker f$ *such that*

$$||u_0 - \hat{u}||_{L^2} = \inf_{u \in \ker f} ||u_0 - u||_{L^2}.$$

In fact, consider cu_0 *if necessary one may assume that* $\inf_{u \in \ker f} ||u_0 - u||_{L^2} = 1$. *Take a sequence* $u_n \in \ker f$ *such that*

$$\lim_{n \to \infty} ||u_0 - u_n||_{L^2} = 1.$$

Put

$$v_n := u_0 - u_n, \quad \lim_{n \to \infty} ||v_n||_{L^2} = 1.$$

then $||\frac{v_n + v_m}{2}||_{L^2} \geq 1$ *and the following identity*

$$||\frac{v_n - v_m}{2}||_{L^2}^2 = \frac{1}{2} \left(||v_n||_{L^2}^2 + ||v_m||_{L^2}^2 \right) - ||\frac{v_n + v_m}{2}||_{L^2}^2 \tag{0.1}$$

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gives

$$||\frac{v_n - v_m}{2}||_{L^2}^2 \leq \frac{1}{2}\left(||v_n||_{L^2}^2 + ||v_m||_{L^2}^2 - 2\right) \to 0 \ \ (n, m \to \infty)$$

Hence $\{v_n\}$ is a Cauchy sequence and has a limit, say $\hat{v}$. One may check (try!) that $\hat{u} = u_0 - \hat{v}$.

Fact 3 (try). *There exists a unique $u_f \in L^2$ such that $f(u_f) = ||u_f||_{L^2}^2$ and*

$$||u_f||_{L^2} = \inf_{u \in \ker f} ||u_f - u||_{L^2}.$$

Such u_f satisfies

$$\int_0^1 u_f(x)u(x)\, dx = 0, \ \ \forall\, u \in \ker f,$$

and one may further check that u_f satisfies $(\star)$.

The key in the above Facts is (0.1), which suggests to introduce the following

Definition. *A Banach space E is said to be uniformly convex if for every $0 < \varepsilon < 2$*

$$\sup\left\{||\frac{x+y}{2}|| : ||x|| \leq 1, \ \ ||y|| \leq 1, \ \ ||x - y|| \geq \varepsilon, \ x, y \in E\right\} < 1.$$

Fact 2 can be generalized to all uniformly convex spaces.

Theorem 1. *Let A be a non-empty closed convex set in a uniformly convex space E. Then for every $u_0 \in E \setminus A$ there exists a unique $\hat{u} \in A$ such that*

$$||u_0 - \hat{u}|| = \inf_{u \in A} ||u_0 - u||.$$

Proof. Similar to the proof of Fact 2, we leave it to the reader. □

Let us apply the above theorem to the case that $E = L^p, \ A = \ker f, \ f(u_0) \neq 0$, where f is a (non-trivial) bounded ($\mathbb{R}$-)linear functional on L^p. Assume that L^p is uniformly convex (one may prove this using the Clarkson's inequality, see page 95 of the Brezis book or the its simplification — Hanner's inequality in [1, Theorem 2.5]), then the above theorem gives

Fact 2'. *There exists a unique $\hat{u} \in \ker f$ such that*

$$||u_0 - \hat{u}||_{L^p} = \inf_{u \in \ker f} ||u_0 - u||_{L^p}.$$

Put

$$u_f := \frac{f(u_0 - \hat{u})}{||u_0 - \hat{u}||_{L^p}^2}(u_0 - \hat{u}),$$

we can prove that

Lemma. $f(u_f) = ||u_f||_{L^p}^2 = ||f|| \cdot ||u_f||_{L^p}$.

Proof. The definition of u_f directly gives

$$f(u_f) = ||u_f||^2_{L^p}.$$

Hence it suffices to check that $||u_f||_{L^p} = ||f||$. Note that

$$||f|| = \sup_{u\in L^p} \frac{|f(u)|}{||u||_{L^p}} = \frac{||u_f||^2_{L^p}}{\inf_{u\in L^p, f(u)=f(u_f)} ||u||_{L^p}} = \frac{||u_f||^2_{L^p}}{\inf_{u\in \ker f} ||u_f - u||_{L^p}},$$

hence the lemma follows directly

$$||u_f||_{L^p} = \inf_{u\in \ker f} ||u_f - u||_{L^p}.$$

□

Similar to Fact 3, the above u_f is unique. This uniqueness will direct give an L^q representation formula for f. In fact, assume that

$$f(u) = \int_0^1 u(x)v(x)\, dx$$

for some $v \in L^q$, put

$$v_f = \frac{f(v|v|^{q-2})}{||v|v|^{q-2}||^2_{L^p}} v|v|^{q-2} = \frac{||v||^q_{L^q}}{||v||^{\frac{2q}{p}}_{L^q}} v|v|^{q-2}$$

then

$$||v_f||_{L^p} = \frac{||v||^q_{L^q}}{||v||^{\frac{q}{p}}_{L^q}} = ||v||_{L^q} = ||f||.$$

So the uniqueness gives

$$u_f = v_f,$$

which implies the following theorem.

Theorem 2. *Let f is a (non-trivial) bounded ($\mathbb{R}$-)linear functional on L^p. Then there exists a unique $u_f \neq 0$ satisfying $f(u_f) = ||u_f||^2_{L^p}$ and*

$$||u_f||_{L^p} = \inf_{u\in \ker f} ||u_f - u||_{L^p}.$$

Moreover, we have $||f|| = ||u_f||_{L^p}$ and

$$f(u) = \int_0^1 u(x)v(x)\, dx, \quad v(x) := ||f||^{\frac{q-p}{q}} \frac{u_f}{|u_f|^{\frac{q-2}{q-1}}}.$$

Remark. *The proof in our textbook of $(L^p)^* = L^q$ depends on the measure theory. In the Brezis book, the main idea is to use convex analysis, more precisely, the fact that every uniformly convex space is reflexive. The third approach (also using convex analysis, see* [1, Lemma 2.8,Theorem 2.14]*) is to prove that the functional f satisfying the Lemma is also unique, i.e. for every non zero $u \in L^p$, there is a unique $f \in (L^p)^*$ such that*

$$f(u) = ||f|| \cdot ||u||_{L^p} = ||u||^2_{L^p}.$$

This uniqueness property is a consequence of the fact that $(L^p)^*$ *is uniformly convex, which is equivalent to that* L^p *is uniformly smooth (see wikipedia for the definition of uniform smoothness and* [1, Theorem 2.6] *for a proof of the uniformly smoothness of* L^p*, which is the key in the proof of the convex projection lemma* [1, Lemma 2.8]*).*

REFERENCES

[1] E. H. Lieb and M. Loss, *Analysis*, second ed., vol. 14 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2001. pages 2, 3, 4