Hahn-Banach Theorems

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This will be around the Chapter 3 of the textbook. We will mainly follow "Chapter 1 of Haim Brezis's book *Functional Analysis, Sobolev Spaces and Partial Differential Equations*".

Week 7 will be on a small project "$(L^p)^* = L^q$". Week 8 will be around the Chapter 4 of the textbook.

In the last lecture we proved the finite dimensional Hahn-Banach Lemma. In this note, we shall see how to look at it as an extension property, which can be easily generalized to the infinite dimensional case. First, let use recall the following equivalent description (try!) of our finite dimensional Hahn-Banach Lemma.

Finite dimensional Hahn-Banach Lemma 2. *Let A be a closed convex set in $\mathbb{R}^N$ with $0 \in A$. If $x_0 \notin A$ then there is a linear function l on $\mathbb{R}^n$ such that*

$$l(x_0) > 1 \geq \sup_{x \in A} l(x),$$

Remark. *Let us introduce the following Minkowski functional (also called gauge) of A*

$$p(x) := \inf\{\alpha > 0 : \ \alpha^{-1}x \in A\}.$$

It is not hard (try) to show that $0 \leq p \leq \infty$ is a function on $\mathbb{R}^n$ satisfying

$$p(\lambda x) = \lambda p(x), \ \ p(x+y) \leq p(x) + p(y), \ \ \forall\, x, y \in \mathbb{R}^n, \ \lambda > 0.$$

Moreover, since A is closed, we have that

$$A = \{x \in \mathbb{R}^n : p(x) \leq 1\}.$$

Hence we can rephrase the condition $x_0 \notin A$ as $p(x_0) > 1$. Consider the linear function l_0 defined on the linear subspace $G := \mathbb{R}x_0$ by

$$l_0(tx_0) = tp(x_0),$$

we know that $l_0 \leq p$ on G. If we know that l_0 extends to a linear function, say l, on $\mathbb{R}^n$ with

$$l(x) \leq p(x), \ \forall\, x \in \mathbb{R}^n,$$

then we must have

$$l(x_0) = l_0(x_0) > 1 = \sup_{x \in A} p(x) \geq \sup_{x \in A} l(x),$$

which gives the above Hahn-Banach Lemma.

Finite dimensional Hahn-Banach Lemma 3. *Let $p : \mathbb{R}^n \to \mathbb{R}$ be a function satisfying*

$$p(\lambda x) = \lambda p(x), \ \ p(x+y) \leq p(x) + p(y), \ \ \forall\, x, y \in \mathbb{R}^n, \ \lambda > 0.$$

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Let G be a linear subspace of $\mathbb{R}^n$. Then every linear function g on G with $g \leq p$ on G extends to a linear function f on $\mathbb{R}^n$ with $f \leq p$ on $\mathbb{R}^n$.

Proof. Assume that $G \neq \mathbb{R}^n$, by induction (try!) it suffices to extend g to $G \oplus \mathbb{R}\{x_0\}$ for some $x_0 \notin G$. If such an extension, say f, exists then we should have

$$f(x + tx_0) = g(x) + tf(x_0) \leq p(x + tx_0), \ \ \forall \, x \in G, t \in \mathbb{R}.$$

It suffices to check that there exists a real number $f(x_0)$ such that

$$B \leq f(x_0) \leq A, \ \ A := \inf_{x \in G, t > 0} \frac{p(x + tx_0) - g(x)}{t}, \ B := \sup_{x \in G, t < 0} \frac{p(x + tx_0) - g(x)}{t}.$$

Hence we need to verify that $B \leq A$, which is equivalent to that

$$\frac{p(y - sx_0) - g(y)}{-s} \leq \frac{p(x + tx_0) - g(x)}{t}, \quad \forall x, y \in G, t, s > 0,$$

since g is linear and $p(\lambda x) = \lambda p(x)$ for positive λ, this is equivalent to

$$g\left(\frac{y}{s} + \frac{x}{t}\right) \leq p\left(\frac{x}{t} + x_0\right) + p\left(\frac{y}{s} - x_0\right),$$

which follows directly from

$$p\left(\frac{x}{t} + x_0\right) + p\left(\frac{y}{s} - x_0\right) \geq p\left(\frac{y}{s} + \frac{x}{t}\right) \geq g\left(\frac{y}{s} + \frac{x}{t}\right)$$

since $p(a) + p(b) \geq p(a + b)$ and $p \geq g$ on G. □

Assuming Zorn's Lemma one may prove the following general Hahn-Banach Theorem.

Hahn-Banach Theorem. *Let E be a vector space over $\mathbb{R}$. Let $p : E \to \mathbb{R}$ be a function satisfying*

$$p(\lambda x) = \lambda p(x), \ \ p(x + y) \leq p(x) + p(y), \ \ \forall \, x, y \in \mathbb{R}^n, \ \lambda > 0.$$

Let G be a linear subspace of $\mathbb{R}^n$. Then every linear function g on G with $g \leq p$ on G extends to a linear function f on $\mathbb{R}^n$ with $f \leq p$ on $\mathbb{R}^n$.

Proof. See Chapter 3 of our textbook or Chapter 1.1 of the Brezis book. □

In the lecture, we will do Corollary 1.2, 1.3, 1.4 and Section 1.3 of the Brezis book.