Finite dimensional Banach spaces and their duals

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Week 4: we will use this note to replace Chapter 2 of our textbook.

Week 5, 6 will be around Chapter 3, but we will mainly follow "Chapter 1 of Haim Brezis's book *Functional Analysis, Sobolev Spaces and Partial Differential Equations*".

Week 7, 8 will be around Chapter 4. Another reference is "Chapter 2 of Brezis's book".

In the last lecture we proved the following result.

All norms on $\mathbb{R}^n$ are equivalent. *Let $||\cdot||$ be a norm on $\mathbb{R}^n$. Then there exists a constant $C > 0$ such that*

$$C^{-1}|x|^2 \leq ||x|| \leq C|x|, \ \ \forall\, x \in \mathbb{R}^n,$$

where $|x| := \sqrt{x_1^2 + \cdots + x_n^2}$ is the Euclidean norm.

Recall that the *dual space* X^* of a normed space X is the space $\mathcal{L}(X, \mathbb{R})$ of *bounded* linear functionals on X. We know that

$$||x^*|| := \sup_{x \in X} \frac{|x^*(x)|}{||x||}$$

defines a norm on X^*. In our case, all linear functionals on $\mathbb{R}^n$ have the following form

$$x^*(x) = c \cdot x := c_1 x_1 + \cdots + c_n x_n.$$

It is clear that all such x^* are bounded. Later we shall identify x^* with a vector $c \in \mathbb{R}^n$, and call

$$||c||_* := ||x^*|| = \sup_{x \in X} \frac{|c \cdot x|}{||x||}$$

the *dual norm* of $||\cdot||$. In this note, we will study the relation between the norms and the dual norms and compare it with the duality relation in convex analysis. The key concept behind these two dualities is the notion of *Legendre transform*.

Legendre transform. *The Legendre transform of a function $\phi : \mathbb{R}^n \to [-\infty, \infty]$ is another function on $\mathbb{R}^n$ defined by*

$$\phi^*(c) := \sup_{x \in \mathbb{R}^n} \{c \cdot x - \phi(x)\}.$$

Example 1. *When $\phi(x) = ||x||$ we have*

$$\phi^*(c) := \sup_{x \in \mathbb{R}^n} \{c \cdot x - ||x||\}, \ \textit{later we will write it as} \ \sup_{x \in \mathbb{R}^n} c \cdot x - ||x||.$$

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We shall prove in the lecture (try!) that

$$\phi^*(c) = \begin{cases} 0 & ||c||_* \leq 1 \\ \infty & ||c||_* > 1, \end{cases}$$

in particular ϕ^* is determined by the dual norm. So we know that the Legendre transform of the norm function is described by the dual norm. In general, we have the following result.

Proposition 1. *Let f be a non negative function on $\mathbb{R}$ with $f(0) = 0$. Then*

$$(f \circ ||\cdot||)^* = f^* \circ ||\cdot||_*.$$

Proof. Write $(f \circ ||\cdot||)^*$ as ψ then

$$\psi(c) = \sup_{x \in \mathbb{R}^n} c \cdot x - f(||x||) = \sup_{t \geq 0} \sup_{||x||=t} c \cdot x - f(t) = \sup_{t \geq 0} t||c||_* - f(t).$$

Since $f \geq 0$, we know that if $t \leq 0$ then

$$t||c||_* - f(t) \leq 0 \leq t||c||_* \leq 0 = 0||c||_* - f(0),$$

which implies

$$\sup_{t \geq 0} t||c||_* - f(t) = \sup_{t \in \mathbb{R}} t||c||_* - f(t) = f^*(||c||_*).$$

The proof is complete. ∎

Example 2. *Take $f(t) = \frac{|t|^p}{p}$, here $p > 1$ is a constant, try to show that*

$$f^*(s) = \frac{|s|^q}{q}, \quad \text{where } q > 1 \text{ is determined by } \frac{1}{p} + \frac{1}{q} = 1.$$

Notice that we always have

$$f(t) + f^*(s) \geq ts$$

from the definition of the Legendre transform. So this example gives the following Hölder type inequality

$$\frac{|t|^p}{p} + \frac{|s|^q}{q} \geq ts.$$

(try to use it to prove the Hölder inequality for integrals — Hint, assume first that $\int |f|^p = \int |g|^p = 1$)

$$\left| \int_0^1 f(x)g(x)\,dx \right| \leq \left(\int_0^1 |f(x)|^p\,dx \right)^{\frac{1}{p}} \left(\int_0^1 |g(x)|^q\,dx \right)^{\frac{1}{q}}, \quad \forall f, g \in C[0,1].$$

The proof in Example 2 also implies that $(f^*)^* = f$ in case $f(t) = \frac{|t|^p}{p}$. In general, it is natural to ask when do we have $(\phi^*)^* = \phi$ for a function ϕ on $\mathbb{R}^n$. An obvious necessary condition is that ϕ is the supremum of a family of affine functions. The following result implies that this is also a sufficient condition.

Theorem 1 (Fenchel-Moreau). *$\phi : \mathbb{R}^n \to [-\infty, \infty]$ satisfies $(\phi^*)^* = \phi$ if and only if ϕ is a lower semi-continuous convex function or ϕ is equal to $-\infty$ identically.*

Recall that ϕ is said to be lower semi-continuous if $\phi(x_0) \leq \liminf_{0<|x-x_0|\to 0} \phi(x), \ \forall\, x_0 \in \mathbb{R}^n$. ϕ is said to be convex if $\phi > -\infty$ and

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y), \ \forall\, x, y \in \mathbb{R}^n, \ \ 0 < t < 1.$$

Note that if ϕ is convex then $\{\phi < \infty\}$ is a convex set (can be non-closed and non-open).

Proof of Theorem 1. Since supremums of a family of linear functions are lower semi-continuous convex or equal to $-\infty$ identically, it suffices to show that if ϕ is lower semi-continuous convex then $\phi = (\phi^*)^*$. First note that (assume that ϕ is not always equal to ∞)

$$\phi(x) + \phi^*(c) \geq c \cdot x$$

implies $\phi \geq (\phi^*)^*$. Hence it suffices to show that $\phi(x_0) \leq (\phi^*)^*(x_0)$ for every $x_0 \in \mathbb{R}^n$.

Step 1: Show that $A_\phi := \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : \phi(x) \leq t\}$ is a closed convex set in $\mathbb{R}^{n+1}$ if ϕ is lower semi-continuous convex.

Step 2: Prove the Hahn-Banach Lemma below.

Step 3: For every $t_0 < \phi(x_0)$, we know that $(t_0, x_0) \notin A_\phi$. Since A_ϕ is closed by Step 1, there exists a small closed ball B around (t_0, x_0) such that

$$B \cap A_\phi = \emptyset.$$

Apply the Hahn-Banach Lemma below to the case that

$$A = A_\phi - B := \{P - Q : P \in A_\phi, \ \ Q \in B\}, \ \ x_0 = 0,$$

we know that there exists a hyperplane separating A_ϕ and B. Since $(t_0, x_0) \in B$, that hyperplane must be equal to the graph of an affine function, say

$$l(x) = c_0 \cdot (x - x_0) + a_0$$

with $a_0 = l(x_0) > t_0$. Since l lies below A_ϕ, we have

$$\phi(x) \geq c_0 \cdot (x - x_0) + a_0.$$

Hence by definition

$$\phi^*(c_0) \leq c_0 \cdot x_0 - a_0$$

and

$$(\phi^*)^*(x_0) \geq c_0 \cdot x_0 - \phi^*(c_0) \geq a_0 > t_0.$$

When t_0 goes to $\phi(x_0)$ the above estimate gives $(\phi^*)^*(x_0) \geq \phi(x_0)$. □

Hahn-Banach Lemma. *Let A be a non-empty closed convex set in $\mathbb{R}^N$. If $x_0 \notin A$ then there is an affine function l on $\mathbb{R}^n$ such that*

$$l(x_0) > \sup_{x \in A} l(x).$$

Proof. Since A is closed, one may take $\hat{x} \in A$ such that

$$|x_0 - \hat{x}| = \inf_{x \in A} |x_0 - x|.$$

Take

$$l(x) = (x_0 - \hat{x}) \cdot (x - x_0).$$

we claim that

$$l(x_0) = 0 > -|x_0 - \hat{x}|^2 = l(\hat{x}) = \sup_{x \in A} l(x).$$

Otherwise, there would exist $x_1 \in A$ such that $l(x_1) > l(\hat{x})$, i.e.

$$(x_0 - \hat{x}) \cdot (x_1 - \hat{x}) > 0.$$

Put

$$f(t) := |\hat{x} + t(x_1 - \hat{x}) - x_0|^2.$$

Since A is convex, we have

$$\hat{x} + t(x_1 - \hat{x}) \in A, \ \ \forall\, 0 \leq t \leq 1,$$

thus

$$f(t) \geq f(0), \ \ \forall\, 0 \leq t \leq 1.$$

But note that

$$f'(0) = -(x_0 - \hat{x}) \cdot (x_1 - \hat{x}) < 0$$

gives $f(t) < f(0)$ when $t > 0$ is sufficiently small, so we get a contradiction. □

Theorem 2 (Fenchel-Rockafellar). *Take two functions $\phi, \psi : \mathbb{R}^n \to (-\infty, \infty]$. Assume that*

$$a := \inf_{x \in \mathbb{R}^n} \phi(x) + \psi(x)$$

is finite and there exists $x_0, c_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$ such that

$$a - \psi(x) \leq c_0 \cdot (x - x_0) + t_0 \leq \phi(x), \ \ \forall\, x \in \mathbb{R}^n.$$

Then

$$a = \sup_{c \in \mathbb{R}^n} -\phi^*(c) - \psi^*(-c) = -\phi^*(c_0) - \psi^*(-c_0).$$

Proof. Note that

$$\phi(x) + \psi(x) + \phi^*(c) + \psi^*(-c) \geq x \cdot c - x \cdot c = 0$$

directly gives

$$a \geq \sup_{c \in \mathbb{R}^n} -\phi^*(c) - \psi^*(-c).$$

Hence it suffices to show that

$$a \leq -\phi^*(c_0) - \psi^*(-c_0).$$

Note that our assumption directly gives

$$\phi^*(c_0) \leq c_0 \cdot x_0 - t_0$$

and

$$\psi^*(-c_0) \leq -c_0 \cdot x_0 + t_0 - a.$$

Hence we get

$$-\phi^*(c_0) - \psi^*(-c_0) \geq a$$

and the proof is complete. □

Remark. *The assumption in the above theorem means there exists an affine function l such that*

$$a - \psi \leq l \leq \phi.$$

Since we already know that $a - \psi \leq \phi$. Intuitively this assumption should hold for convex ϕ and ψ (with certain regularity assumptions, see Theorem 1.12 in the Brezis book).

Exercise 1. *Show that*

$$||f||_p := \left(\int_0^1 |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

defines a norm on $C[0,1]$ when $p \geq 1$. Fix $g \in C[0,1]$, show that

$$\sup_{f \in C[0,1]} \frac{|\int_0^1 f(x)g(x)\,dx|}{||f||_p} = \begin{cases} \sup_{x\in[0,1]} |g(x)| & p = 1 \\ \left(\int_0^1 |f(x)|^q \, dx \right)^{\frac{1}{q}} & p > 1, \end{cases}$$

where $q > 1$ is determined by $\frac{1}{p} + \frac{1}{q} = 1$.

Exercise 2. *Show that $\phi : \mathbb{R}^n \to [-\infty, \infty]$ satisfies $\phi^* = \phi$ if and only if $\phi(x) = |x|^2/2$.*

Exercise 3 (hard exercise, try!). *Let ϕ, ψ be two smooth convex functions with $-\psi \leq \phi$ on $\mathbb{R}^n$. Show that there exists an affine function l such that $-\psi \leq l \leq \phi$ on $\mathbb{R}^n$.*