TMA 4230 - SPRING 2020

Topics for the exam

1. Review of normed spaces and Banach spaces. Dual spaces. Cauchy sequences
2. c_0, l_p, $L_p(\mu)$, $C[0,1]$ and their duals.
3. Zorn's lemma. Hamel basis. Construction of unbounded linear functional on an infinite dimension Banach space.
4. Sublinear functionals and Hahn-Banach theorem for real Banach spaces. Hahn-Banach theorem for complex Banach spaces.
5. Applications of Hahn-Banach theorem. l_∞^*-space. Bidual of a Banach space, natural embedding $X \subset X^{**}$. Reflexive spaces.
6. Schauder basis, separable Banach spaces.
7. New Banach spaces from old: direct sum of Banach spaces, quotient spaces X/Y.
8. Isomorphism and isometric isomorphism of Banach spaces. The adjoint of an operator.
9. Baire category theorem.
10. Uniform boundedness principle. Banach-Steinhaus theorem.
11. Opens Mapping Theorem and Bounded Inverse Principle.
12. Closed Graph theorems.
13. Density of trigonometric polynomials in $L_1(\mathbb{T})$. Riemann-Lebesgue Lemma. Existence of a sequence in $c_0(\mathbb{Z})$ which not the Fourier transform of a function in $L_1(\mathbb{T})$.
14. Banach-Mazur characterization of separable spaces.
15. Topological vector spaces. Dual spaces. Strong and weak topologies on X. Weak$*$ topology on X^*.
16. Compact sets in a topological vector spaces. Tychonoff's theorem (without proof). Banach-Alaoglu theorem. Goldstine's theorem (without proof).
17. Eigenvector's of an operator. Spectrum and resolvent set. Compact operators.
18. Rank-nullity theorem for compact operators, integral operators, Hilbert-Schmidt operators, Fredholm operators.
19. Spectral theory of compact operators.
20. Basics of Hilbert spaces. Riesz representation theorem.

Problems and questions

1. Prove that if X and Y are Banach spaces and $f : X \to Y$ is a linear operator then f is bounded if and only if f is continuous.
2. Prove that the space $\mathcal{L}(X, Y)$ of all bounded linear operators from normed space X to Banach space Y is a Banach space equipped with the usual operator norm

$$\|f\|_{\mathcal{L}(X,Y)} = \sup_{\|x\|_X=1} \|f(x)\|_Y.$$

3. Prove that dual space to l_p for $1 \leq p < \infty$ is l_q, where $1/p + 1/q = 1$. Show that l_p for $1 \leq p \leq \infty$ are Banach spaces.
4. Prove that $L_p(\mu)^* = L_q(\mu)$ for $1 \leq p < \infty$ and that $L_p(\mu)$ are Banach spaces for $1 \leq p \leq \infty$.
5. Prove that space of sequences c_0 with the standard norm

$$\|x\| = \sup_{n\in\mathbb{N}} |x_n|$$

is a Banach space. Prove that $c_0^* = l_1$.
6. Prove that space of convergent sequences c with the standard sup norm is a Banach space.
7. Let X be a Banach space on $\mathbb{R}$ with $\dim(X) = +\infty$. Use Hamel basis to construct an unbounded linear functional $f : X \to \mathbb{R}$.
8. Use one-dimensional Hahn-Banach theorem to prove that for every $x \in X$ with $\|x\| = 1$ there exists $f \in X^*$ such that $f(x) = \|f\|$. Give an example of such X and $x \in X$ when f is not unique.
9. Prove that the functional $f(x) = \lim_{n\to\infty} x_n$ belongs to c^* and find its norm.
10. Use Hanh-Banach theorem to prove that $l_\infty^* \neq l_1$.
11. Use Baire category theorem to prove that $\mathbb{Q}$ is not a G_δ set.
12. Let X be a Banach space and suppose E is a dense linear subspace whci is G_δ set. Show that $E = X$.
13. Give an example of two Banach spaces X and Y and linear bounded $T : X \to Y$ such that $ran(T) \neq Y$, and $ran(T)$ is dense in Y.
14. Determine the quotient space c/c_0.
15. Prove that the space of functions $L_p(0, 1)$, $0 < p < 1$, is metrizable. To do this show that $d(f, g) = \|f - g\|_p^p$ is a metric. Therefore one can consider $L_p(0, 1)$ as a topological vector space, where topology is given by the metric. Prove that $L_p(0, 1)^* = \emptyset$ for $0 < p < 1$.

Lectures and exercise sessions

March 13

1. Read the proof of Banach-Mazur characterization of separable Banach spaces on p.74-75. The most interesting point is to understand why T_0 preserves norms.
2. Try to solve the following problems: 4.18, 4.19

3. A nice measure theory problem: 4.16.

March 16-March 20

1. Read about topological vector spaces on p.86-87.
2. Read the proof of fact that $L_p(0,1)^* = \emptyset$ for $0 < p < 1$.
Note that this is impossible for a Banach space, where the dual is always non-empty (by Hahn-Banach Theorem).
3. Read the definitions of norm topology and weak topology on a Banach space X. Weak topology is always weaker (contains less open sets) than norm topology if $dim X = +\infty$.
4. Read the definition of weak* topology on X^*.
5. Show that weak topology on X^* is the same as weak* topology only for reflexive spaces X. Otherwise weak* topology is weaker.
6. Show that all the norm on a finite-dimensional vector space are equivalent.
7. Solve the problems 5.1, 5.13, 5.14.

March 23-March 27

1. Read general definition of compact sets in a topological vector spaces on p.85.
2. Tychonoff's theorem. Banach-Alaoglu theorem. Goldstine's theorem.
3. Problem set 7.

March 30-April 3

1. Recall that $x_n \to x_0$ weakly iff $f(x_n) \to f(x_0)$ for all $f \in X^*$. In particular, $e_n \to 0$ weakly in l_p for $1 < p < \infty$. What happens if $p = 1$ or $p = \infty$?
2. Solve the problem 5.25.
3. In fact the following more general statement is true: If $x_n \to 0$ weakly then x_n is bounded. (Hint: Use Uniform Boundedness Principle).
Proof can be founded here.
4. The image of a compact set by continuous map is compact. Therefore if K is compact subset of a Banach space over $\mathbb{R}$ then $f(K)$ is closed in $\mathbb{R}$ for every $f \in X^*$.
5. Banach-Alaoglu theorem claims that closed unit ball B_{X^*} is compact in weak* topology on X^*. However this is not imply that B_X is always compact in weak topology on X, see Problem 5.34 for an example.
6. Compare definitions of compactness and sequential compactness in a topological space. They are the same in metric spaces and also in weak topology of a normed space, see Eberlein-Smulian Theorem

April 13-April 17

During this section we will study eigenvectors and spectrum of bounded linear operators on a Banach space X. We say that λ is an eigenvalue of a bounded linear $T : X \to X$ iff there is $x \neq 0$ such that $Tx = \lambda x$. Also, we say that $\lambda \in Sp(T)$ (that is λ belongs to

spectrum of T) if $T - \lambda I$ is not a bijection, where I is an identity operator. The last is equivalent to the fact that $T - \lambda I$ has no bounded inverse operator. Clearly, if λ is an eigenvalue, then $\lambda \in Sp(T)$. If X is a Banach space over $\mathbb{C}$ and $\dim X < +\infty$, then we are in the usual matrix situation, therefore $\lambda \in Sp(T)$ iff λ is an eigenvalue. However, when $\dim X = +\infty$ it is not always the case.
1. Solve the following problem: Let $X = C[0,1]$ with the standard maximal norm over $\mathbb{C}$ and $T : X \to X$ is given by $Tf(x) = xf(x)$. Prove that T has no eigenvectors. Find spectrum of T. The following simple lemma might be useful:
If $\|A\| = c < 1$, then $I - A$ is invertible and

$$(I - A)^{-1} = I + A + A^2 + ... + .$$

2. Prove that if $|\lambda| > \|T\|$, then $\lambda \notin Sp(T)$.
3. Solve Problems 6.1 and 6.2.

April 20-April 24

1. Read the definition of compact operators on p. 131. For compact operators, all point of spectrum are eigenvalues (except possibly 0), see Rank Nullity Theorem for more precise statement on p. 145.
2. Read examples 6.8-6.10 on p.131.
3. Read examples 6.18, 6.19, 6.21.
4. Read propositions 6.20, 6.22.
5. Read Rank-Nullity Theorem.
6. Solve problems 6.1, 6.2, 6.4, 6.14.

April 27- May 1

1. Recall that if $T : X \to Y$ is a linear bounded operator such that $\dim ran(T) < +\infty$, then T is compact.
2. Also the set $\mathcal{K}(X, Y)$ of all compact operators is a closed linear subspace of $\mathcal{L}(X, Y)$, see Theorem 6.14. Hence T is compact when for every $\varepsilon > 0$ there exists T_ε such that $\|T - T_\varepsilon\| < \epsilon$ and $\dim ran(T_\varepsilon) < +\infty$. This is a very powerful tool to prove compactness of operators. First example of a compact operator that can not be approximated by finite dimensional operators was constructed by Per Enflo only in 1973, see p.139-140 for discussion.
3. Read the proof that Fredholm operators are compact, see proposition 6.23.
4. Read the proof that adjoint of a compact operator is compact, see Theorem 6.26.
5. Read the article Spectral theory of compact operators.
6. Solve problem 6.13.

May 4-May 7

1. Read Section 7.1.
2. Prove that Hilbert-Schmidt operators are compact, see p.140 for the definition.
3. Solve problems 7.1, 7.19.