

## Inner product spaces. Hilbert spaces.

### Def (V1.4.1) Inner product space

Inner product -  $1\frac{1}{2}$ -linear form

- Main model:  $\mathbb{C}^n$
- More examples:  $\ell^2$ ,  $L^2(\mu)$
- Cauchy-Schwarz inequality (V, Th. 1.4.5)
- Cauchy-Schwarz as special case of Hölder.
- Norm in inner product space (V 1.4.6)
- Orthogonality, Pythagorean thm., angle between vectors,
- Two more examples:
  - $m \times n$  matrices, Hilbert-Schmidt norm
  - $L^2(\Omega, \Sigma, \mathbb{P})$
- Polar form - reconstruction  $\langle \cdot, \cdot \rangle$ , through  $\| \cdot \|$ .  
(I think misprint in Vershynin 1.4.19)
- Exercises: Parallelogram law.

## Hilbert spaces

- Definition: inner product + completeness
- Previous examples work
- Def: (Closed linear subspace),
- Examples
- Linear subspace, generated by a set.
- Orthogonal complements. This is also a closed subspace.
- Distance between a vector and closed subspace.
- Vector, delivering this distance: (V.5.5).
  - existence
  - uniqueness
  - orthogonality.
- Orthogonal projection.
- Orthogonal decomposition.
- Notations:  $\mathcal{Y} \supseteq \mathcal{Y}^\perp$ ,  $X = Y \oplus Y^\perp$

## Orthonormal sequences.

- Orthogonal system, Orthonormal system.
- Example Fourier series in  $L^2(-\pi, \pi)$ .
- $\{x_k\} \subset X$ -orthonormal. Series of the form  $\sum a_n x_n$ :
  - Distance between  $\text{Span}\{x_k\}^N$  and  $x \in X$ . (Projection)
- Complete orthonormal systems.
  - Bessel
  - Parseval
- Fourier series.
- Gram-Schmidt orthogonalization.
- Separable spaces. Existence of orthonormal basis.  
All separable Hilbert spaces are isometric.
- Riesz representation theorem.

More examples:

- Chopping the real axis.
- Haar systems. A couple of words about wavelets.

- Chebyshev polynomials.
- A couple of words about general orthonormal polynomials.
- Hermite polynomials.