

Lecture 04 Crash course on measure theory (Appendix A in Br-Kal. book)

1. Measurability

- Definition of σ -algebra
 - .. operations
 - .. examples
 - .. Borel algebra in a metric space.

(X, \mathcal{F}) -measurable space, $A \in \mathcal{F}$
↑ measurable set.

Def: Measurable function $f: X \rightarrow \mathbb{C}$

- operations
- indicator function
- simple function
- Lebesgue (level) set.

Theorem Approximation of positive measurable functions.

2. Positive measures and integration

Def measure (X, \mathcal{F}, μ) .

Borel σ -algebra in \mathbb{R} . (and \mathbb{R}^n).

Borel measure

Negligible sets

Lebesgue measure.

More examples:

- Probability space.
- Point measure in \mathbb{R}^2
- Counting measure.

Construction of the integral (X, \mathcal{F}, μ)

- integral of simple function
- integral of positive function
- Lebesgue integrable functions
- Integral of integrable function.

Convergence Theorems and Fatou's Lemma.

- almost everywhere defined functions.
- equivalence classes of functions.
- μ -measurable: Ignore sets of zero measure.
- almost everywhere convergence.
- Monotone convergence theorem.
- Dominated convergence theorem.
- Fatou's lemma.
- Tool: Tchebyshov inequality.

f - integrable, $a > 0$

$$\mu(\{x : f(x) > a\}) \leq \frac{1}{a} \int |f(x)| d\mu(x)$$

Exercise

Complex measure and Absolute Continuity.

- Complex measure, signed measure.
- Decomposition: μ -signed measure \Rightarrow

Not in the book

$$\Rightarrow X = X_+ \cup X_-, X_+ \cap X_- = \emptyset, \mu = \mu_+ - \mu_-$$

μ_{\pm} - positive measures on X_{\pm} .

positive measure

- $|\mu|(E)$ as variation and as decomposition.
- σ -finite measures,
- Absolute continuity with respect to a positive measure.
- Radon-Nikodym theorem.
- μ -complex measure $\Rightarrow \exists g(x)$ -measurable such that $|g(x)| = 1$ and $d\mu = g d|\mu|$

!!

- All theorems on passing to the limit can be formulated for complex valued measures.
- Exercise: formulate the theorems.

L^p -spaces (X, \mathcal{F}, μ)

- $L^p(\mu)$, $p \in [1, \infty)$
- Reminder: equivalence classes.
- Definition using $\|\cdot\|_\infty$
- Exercise. • Hölder and Minkowski for $\|\cdot\|_p$
- Spaces L^p are complete.

Proof (completeness) $\{f_m\} \subset L^p(\mu)$ - Cauchy sequence

- We need: $\exists f \in L^p(\mu) : \|f - f_m\|_p \rightarrow 0, m \rightarrow \infty$,
- It suffices to find $\{m_n\}, f \in L^p(\mu) : \|f - f_{m_n}\|_p \xrightarrow{n \rightarrow \infty} 0$
- Choice of m_n : $\|f_j - f_k\|_p < 2, j, k > m_n$
- $f_{m_{n+1}} = (f_{m_{n+1}} - f_{m_n}) + (f_{m_n} - f_{m_{n-1}}) + \dots + (f_{m_1} - f_1) + f_1$

$$= f_1 + \sum_0^n h_k; \|h_k\|_p \leq 2^{-k}$$

- Monotone convergence theorem \Rightarrow

$$\sum_1^m |h_k| \nearrow \varphi \in L^p(\mu)$$

- Now $\sum h_k \rightarrow f \in L^p(\mu)$.

- Convenient tool Simple functions are dense in $L^p(\mu)$

- Dual spaces $(L^p(\mu))^* = L^q(\mu) \quad 1 \leq p < \infty$

- Riesz theorem (proof comes later)

$$(C[0,1])^* = \{ \mu, \mu \text{-Borel measure on } [0,1], |\mu|([0,1]) < \infty \}$$

$$\psi_\mu(f) = \int_0^1 f d\mu; \|\psi_\mu\|_{C[0,1]^*} = |\mu|([0,1]).$$