Lectures 1–3

1. About the course.

2. Main object: Linear space (over $\mathbb{R}$ or $\mathbb{C}$)

3. Examples: (look at Vershynin p. 2 for explanation of notation).
   - $\mathbb{R}^n$, $\mathbb{C}^n$
   - $\mathcal{P}$ – polynomials
   - $C[a,b]$
   - $C^\infty[a,b]$
   - ℓ^1
   - ℓ^∞
   - c, c_0, c_{00}

   . . . . . .

   green corresponds to references in B & K.

4. Definition (1.1) Normed spaces.
   - Norm is an analog of length.
   - How does one introduce norm for (some of) examples above.
   - Various norms in the same space: ℓ_1^N, ℓ_N^2, ℓ_N^∞.

5. Definition (1.2) metric space.
   - Norm generates the metric (opposite is wrong).
   - Digression:

     Introducing metric in $C^\infty[0,1]$.

6. Metric spaces: (1.3) Convergence, Cauchy sequences, complete spaces.

7. Main object (1.6) Banach space.
   • Look through the examples.

8 Linear map between two normed spaces.
   • Finite dimensional analog. Matrices.

9. (1.5) Bounded linear operator. Norm.

10. Continuous mapping between two metric spaces.
   (definition as in the calculus course).

11. Proposition (1.6). Linear operators boundedness v.s. continuity

12. Definition (1.9) Space $\mathcal{L}(X, Y)$

13. Exercises: (X, Y - Banach spaces).

   • $T \in \mathcal{L}(X, Y)$ prove $\|T\| = \inf\{K : \|Tx\| \le K\|x\|, \forall x \in X\}$.

   • Check the triangle inequality for $\mathcal{L}(X, Y)$.

   • Check that $\mathcal{L}(X, Y)$ is complete, i.e.

$\{T_n\} \subset \mathcal{L}(X, Y)$, $\{T_n\}$ - Cauchy sequence $\Rightarrow$

$\Rightarrow \exists\, T \in \mathcal{L}(X, Y) : \|T_n - T\| \to 0$

End of lecture 1

13. Definition (1.13) Linear functional. Dual space.

* * *

Spaces ℓ^p, $1 \le p \le \infty$.

14. "Basic" sequences $e^{(i)} = \{\underbrace{0, 0, \ldots, 0, 1}_{i}, 0, 0, \ldots\}$

15. Definition (2.1) ℓ^p space, $1 \le p \le \infty$.

$$\xi = \{\xi_j\}_1^\infty \Rightarrow \|\xi\|_p := \left(\sum |\xi_j|^p\right)^{1/p}$$

16. Toward the proof that ℓ^p is a Banach space:

- Conjugate exponents: $\frac{1}{p} + \frac{1}{q} = 1$, $q = \frac{p}{p-1}$

- Young inequality:

$$a, b > 0 \Rightarrow a^{\frac{1}{p}} \cdot b^{\frac{1}{q}} \le \frac{1}{p} a + \frac{1}{q} b. \quad (*)$$

"=" if $a = b$.

Proof: $t = \frac{a}{b}$; $(*) \Leftrightarrow \varphi(t) := t^p - a \le 1$, $t > 0$

↑ standard max procedure.

- Hölder inequality:

$$\sum |a_k b_k| < \left(\sum |a_k|^p\right)^{1/p} \left(\sum |b_k|^q\right)^{1/q}$$

Proof: – Assume that sequences are finite.

– Assume $\sum |a_k|^p = 1$, $\sum |b_k|^q = 1$

– Apply Young inequality.

• Minkowski inequality (triangle inequality for ℓ^p).

$$\xi = \{\xi_j\},\ \eta = \{\eta_j\} \in \ell^p \Longrightarrow \|\xi+\eta\|_p \le \|\xi\|_p + \|\eta\|_p.$$

Proof: – Assume $\sum |\xi_j + \eta_j|^p = 1$

$$- \sum |\xi_j + \eta_j|^p \le \sum |\xi_j|\,|\xi_j+\eta_j|^{p-1} +$$

$$+ \sum |\eta_j|\,|\xi_j + \eta_j|^{p-1}$$

– Apply Hölder to the left-hand side.

• Exercises

•• When do we have equality in the Minkowski inequality?

•• Generalised Hölder:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \Longrightarrow \|f\cdot g\|_r \le \|f\|_p \|g\|_q$$ ← Prove

•• Two sided spaces $\{\xi_j\}_{j=-\infty}^{\infty}$

Convolution $(\xi * \eta)_k = \sum_j \xi_j \eta_{k-j}$

Prove $\xi \in \ell^p$, $\eta \in \ell^1 \Longrightarrow \xi * \eta \in \ell^p$ and $\|\xi*\eta\|_p \le \|\xi\|_p \cdot \|\eta\|_1$

• Last step: ℓ^p is complete.

End of lecture 2.

17. Remind notation: $e^i := \{0, \dots, 0, \underbrace{1}_{i}, 0, \dots\}$

$$\xi = \{\xi_j\} \in \ell^p \Rightarrow \xi = \sum_j \xi_j e^j \quad \leftarrow \text{Convergence in } \ell^p.$$

18. Definition: ($\sqrt{}$ p. 79) Shauder basis.

Vershynin

In case a Shauder basis exists, linear operators correspond to infinite matricies

19. Dual to ℓ^p.

Theorem (2.5).

$$1 \le p < \infty \Rightarrow (\ell^p)^* = \ell^q$$

and $\eta = \{\eta_j\} \in \ell^q$, $\xi = \{\xi_j\} \in \ell^p \Rightarrow f_\eta(\xi) = \sum \xi_j \eta_j$

and $\|f_\eta\|_{(\ell^p)^*} = \|\eta\|_q$ ← the corresponding functional.

Proof. $1 < p < \infty$

If $\eta = \{\eta_j\} \in \ell^q \Rightarrow f_\eta : \xi \mapsto \sum \xi_j \eta_j$ is in $(\ell^p)^*$

← by Hölder.

b). Given $\psi \in (\ell^p)^*$ we have to find $\eta = \{\eta_j\} \in \ell^q$ such that $\psi = \psi_\eta$.

✓. $\eta_j := \psi(e^j)$. $\qquad \eta := \{\eta_j\} \overset{?}{\in} \ell^q$

✓ Denote $\zeta_j := |\eta_j|^{\frac{1}{p-1}} e^{-i \arg \eta_j}$, $\quad \zeta := \{\zeta_j\}$.

$$\Rightarrow \sum |\zeta_j|^p = \sum |\eta_j|^q, \quad \|\zeta\|_p = \left(\sum |\eta_j|^q\right)^{1/p} \Rightarrow$$

$$\Rightarrow |\psi(\zeta)| \le \|\psi\| \left(\sum |\eta_j|^q\right)^{1/p}. \qquad (*)$$

For convenience assume all sums to be finite.

✓ $|\psi(\zeta)| = \sum \eta_j \zeta_j = \sum |\eta_j|^q \quad (**)$

✓ Compare $(*)$ and $(**)$ $\Rightarrow \eta \in \ell^q$; $\|\psi\|_{\ell^{p*}} \le \|\eta\|_q$

✓ Exercise: $\|\psi\|_{(\ell^p)^*} = \|\eta\|_q$.

c) $(\ell^1)^* = \ell^\infty$.

d) Remark: $(\ell^\infty) \underset{\neq}{\supset} \ell^1$,

e) $(c_0)^* = \ell^1$.

f) Definition (closed) subspace.

g) Example: $c \supset c_0$.