1. Given a bounded sequence $a = \{a_n\}$ consider the operator $T_a : l^2 \to l^2$, defined by

$$T_a : \{x_n\} \mapsto \{a_n x_n\}.$$

- Prove that T_a is invertible iff $\inf |a_n| > 0$. Find the inverse if exists.

- For which λ the operator $T_a - \lambda I$ is invertible?

2. Let X be a Banach space, $\{x_n\} \subset X$ and $x_n \xrightarrow{w} x$ for some $x \in X$. Prove that $\sup \|x_n\| < \infty$.

3. Let $\mathcal{H}$ be a Hilbert space $\{x_n\} \subset H$, $x_n \xrightarrow{w} x$ for some $x \in \mathcal{H}$ and also $\|x_n\| \to \|x\|$, $n \to \infty$. Prove that $\|x - x_n\| \to 0$.

4. Let M_1, M_2 be two complete metric spaces, M_1 is compact and $f : M_1 \to M_2$ is a continuous mapping. Prove that $f(M_1) \subset M_2$ is a compact set.

5. Consider the closures of the following sets of functions in $C[0,1]$:

- $\{(at)^n\}$, $n = 1, 2, \ldots$

- $\{\sin(t+n)\}$, $n = 1, 2, \ldots$

- $\int_0^1 e^{ts} \varphi(t)\, dt$, $s > 0$,

here φ is a fixed function in $C[0,1]$

which of these sets (if any) are compacts.