Ex. set # 6.

1. Let X be a vector space $\mathbb{R}^2$ with $\|\cdot\|_p$ norm an $V = \{(x,y) \in \mathbb{R}^2 : x - 2y = 0\}$. Define the linear functional on V as $\varphi : (x,y) \mapsto x$.
   - Find the norm of φ
   - What is the norm preserving prolongation of φ on X for $p = 1, 2, \infty$?

2. Let $A \in \mathcal{L}(X,Y)$ and A is invertible, $A^{-1} \in \mathcal{L}(Y,X)$. Prove that A^* is invertible and $(A^{-1})^* = (A^*)^{-1}$.

3. Let $\mathcal{H}$ be a Hilbert space, $A \in \mathcal{L}(\mathcal{H})$. Prove that $\|AA^*\| = \|A\|^2$.

4. Let $X = \ell^1$. Find A^* if
   - $A : (x_1, x_2, \dots) \mapsto (x_1, x_2, \dots, x_n, 0, 0, \dots)$
   - $A : (x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, x_4, \dots)$
   - $A : (x_1, x_2, x_3, \dots) \mapsto (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \dots)$, here $\{\lambda_k\} \in \ell^\infty$ is a given sequence.

5. Let $X = L^2(0,1)$. Find A^* if
   - $Ax(t) = \int_0^t x(\tau)\, d\tau$;
   - $Ax(t) = \int_0^1 f(\tau) x(\tau)\, d\tau$
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