Problem set # 5

1. Let X be a Banach space over $\mathbb{C}$, $X_0 \subset X$ closed subspace and $f \in X_0^*$. Prove that f can be extended to an element in X^* with the same norm.

Hint: Let $f(x) = f_1(x) + i f_2(x)$, $x \in X$. Prove that f_1, f_2 a linear functionals in X if considered as a linear space over $\mathbb{R}$ and also $f_2(x) = i f_1(x)$, $x \in X_0$

Then extend f_1 as a real functional to $\tilde{f}_1$ - real functional in X with $\|\tilde{f}_1\|_{X^*} = \|f_1\|_{X_0^*}$ and set $\tilde{f}(x) = \tilde{f}_1(x) + i\tilde{f}_1(ix)$.

Prove that this is an extension of f and $\|\tilde{f}\| = \|f\|$.

2. Exercise from lecture of 07.02 – item I in lecture notes

3. Let c_0 and c be the spaces of sequences $\xi = \{\xi_k\}_1^\infty$ such that $\lim_{k\to\infty} \xi_k = 0$ and $\lim_{k\to\infty} \xi_k$ exists. We provide them with $\| \ \|_\infty$ norm. Prove that $c_0^* = \ell^1$. Can you describe c^*?

4. Prove that the following functionals are linear bounded functionals in $C[-1,1]$ and find their norms:

- $f(x) = \frac{1}{3}\left[x(-1) + x(1)\right]$, $x \in C[-1,1]$

- $\psi(x) = \frac{1}{3}\left[x(-1) - x(1)\right]$
- $\psi(x) = \int_0^1 x(t)\,dt$
- $\psi(x) = \int_{-1}^{1} t f(t)\,dt$

5. Which of the following functionals in $C[0,1]$ (if any) is bounded:

- $\psi(x) = \int_0^1 x(\sqrt{t})\,dt$
- $\psi(x) = \int_0^1 x(t^2)\,dt$
- $\psi(x) = \lim_{n\to\infty} \int_0^1 x(t^n)\,dt$ $\Leftarrow$ can you find it explicitely ?